BMO 2005–2006 Round 1
Problem 6—Solutions

Joseph Myers

November 2005

Introduction

Problem 6 is:

> 6. Let $\mathcal{T}$ be a set of 2005 coplanar points with no three collinear. Show that, for any of the 2005 points, the number of triangles it lies strictly within, whose vertices are points in $\mathcal{T}$, is even.

This result is true for any odd integer replacing 2005. For all six solutions presented here, all points are always presumed coplanar, and it is natural to distinguish the point P for which the number of containing triangles is being counted from the other 2004 points of $\mathcal{T}$. That is, all solutions prove the following result:

> For any nonnegative even integer n, and any set $\mathcal{S}$ of n points, no three collinear, and any point $P \notin \mathcal{S}$ not collinear with any two points of $\mathcal{S}$, the number of triangles containing P whose vertices are points of $\mathcal{S}$ is even.

I found the given solutions in the order 4, 5, 6, 3, 2; solution 1 was provided by Richard Atkins and Gerry Leversha and is the "standard" approach to this problem. Solution 7 is from Paul Russell; solution 8 is from Georg Schoenherr; solution 10 is from my mother; solutions 9 and 11 are loosely based on ideas from contestant scripts and in turn solution 12 is loosely based on solution 9. I have also adjusted the typography of the problem, changing T to $\mathcal{T}$.

Solution 1: Sliding a point

Consider starting a point X somewhere outside the convex hull of $\mathcal{S}$ and moving it to the desired position of P along some continuous path. Without loss of generality we may choose X and the path such that it is a straight line that does not pass through any of the points of $\mathcal{S}$ or the intersection of any two lines between points of $\mathcal{S}$. Keep track of the number of triangles containing X at all points along this path. Initially it is 0; it only changes when it crosses the straight line segment between two points of $\mathcal{S}$, say A and B. If there are c points of $\mathcal{S}$ on the same side of the line AB as X starts, and d points on the other side (so $c + d = n - 2$), then X leaves c triangles and enters d triangles when it crosses the line. Since $c + d$ is even, so is $d - c$, so the number of triangles containing X remains even at all points along its path.

The choice of a straight line for the path is to avoid it crossing lines infinitely many times. At BMO level I do not think it is to be expected for students to state that their paths must cross lines only finitely many times or to give any reason why such a path must exist. I also avoid intersection points for simplicity, although things can also be properly defined when an intersection point is crossed and students may well omit to consider the issue. The solution can also be phrased in terms of keeping P fixed and moving the points of $\mathcal{S}$, and Geoff Smith points out that you can also move both P and all the points of $\mathcal{S}$.

Solution 2: Direct counting

For any set $\mathcal{A}$ of three points, not collinear, and any point P not collinear with any two points of $\mathcal{A}$, define $I^{\Delta}_{\mathcal{A}}(P)$ to be 1 if P is contained within the triangle whose vertices are the elements of $\mathcal{A}$ and 0 otherwise. For any set $\mathcal{B}$ of points, no three collinear, and any point $P \notin \mathcal{B}$ not collinear with any two points of $\mathcal{B}$, define $I_{\mathcal{B}}(P)$ to be the number of triangles containing P whose vertices are points of $\mathcal{B}$, so $I_{\mathcal{A}}(P) = I^{\Delta}_{\mathcal{A}}(P)$ for a set of three points and in general

$$I_{\mathcal{B}}(P) = \sum_{\mathcal{A}\subseteq\mathcal{B},\, |\mathcal{A}|=3} I^{\Delta}_{\mathcal{A}}(P).$$

Note that $I_{\mathcal{B}}(P)$ is even if $|\mathcal{B}| = 4$, by considering the two possible configurations for four points (either the convex hull is a quadrilateral, or it is a triangle; see illustrations under solution 6 below).

We wish to show that $I_{\mathcal{S}}(P)$ is even. Observe that

$$\sum_{\mathcal{B}\subseteq\mathcal{S},\, |\mathcal{B}|=4} I_{\mathcal{B}}(P)$$

is even, because each individual term of the sum is even. But

$$\begin{aligned}
\sum_{\mathcal{B}\subseteq\mathcal{S},\, |\mathcal{B}|=4} I_{\mathcal{B}}(P) &= \sum_{\mathcal{B}\subseteq\mathcal{S},\, |\mathcal{B}|=4} \ \sum_{\mathcal{A}\subseteq\mathcal{B},\, |\mathcal{A}|=3} I^{\Delta}_{\mathcal{A}}(P) \\
&= (n-3) \sum_{\mathcal{A}\subseteq\mathcal{S},\, |\mathcal{A}|=3} I^{\Delta}_{\mathcal{A}}(P) \\
&= (n-3) I_{\mathcal{S}}(P)
\end{aligned}$$

since each 3-set is a subset of $(n-3)$ 4-sets. Since $n-3$ is odd, $I_{\mathcal{S}}(P)$ is even.

Gerry Leversha shows how this solutions can be written in words without use of indicator functions, which is probably how BMO candidates are more likely to write it.

Solution 3: Rotating a line with binary sequences

Draw a straight line through P, not passing through any point of $\mathcal{S}$. Label one half-line starting at P as 1 and the other as 0. Rotate this line clockwise around P, writing down 1 when the 1-half-line passes through a point of $\mathcal{S}$ and 0 when the 0-half-line does so. When the line has been rotated by 180°, n digits will have been written down, one for each point of $\mathcal{S}$; after this each digit is different from that n before it, so the sequence repeats with period $2n$.

P is *not* contained in the triangle whose vertices are three given points of $\mathcal{S}$ if and only if those points all lie on one side of some line through P; that is, if the subsequence of digits corresponding to those points has pattern $\ldots 111000111000 \ldots$; so it *is* contained in that triangle if and only if the pattern is $\ldots 101010101010 \ldots$. Thus, the number of triangles containing P is equal to the number of 010 and 101 subsequences of (not necessarily consecutive) digits from any n consecutive digits in the sequence written down by the above procedure.

We claim that the number of such subsequences is always even, and prove this by induction on the even integer n. It is true for $n = 0$ and $n = 2$, so suppose $n \geq 4$ and that the number of such subsequences of any sequence of m 0s and 1s is even for any even $m < n$. Describe a sequence of a_1 1s, a_2 0s, a_3 1s, $\ldots$ as $(a_1, a_2, a_3, \ldots)$, where the a_i are nonnegative integers, and say this description is in canonical form if no a_i is zero except possibly a_1. Let $S(a_1, a_2, a_3, \ldots)$ denote the number of 010 and 101 subsequences in this sequence; then

$$S(a_1, a_2, a_3, \ldots) = \sum_{i<j<k,\, j-i, k-j \text{ odd}} a_i a_j a_k$$

independent of whether the description is canonical. But if $a_i \equiv b_i \pmod 2$ for all i then this formula implies that

$$S(a_1, a_2, a_3, \ldots) \equiv S(b_1, b_2, b_3, \ldots) \pmod 2$$

so if any $a_i \geq 2$ then the result is true by the induction hypothesis.

Thus it only remains to consider the case where the canonical description has all a_i zero or one; that is a sequence of form $101010\ldots$ (without loss of generality). Let $n = 2k$. The number of 010 and 101 subsequences not using either of the initial two digits is even by the induction hypothesis. The number using both those digits is $k - 1$. The number using just one of those digits is the number of pairs in the remaining $n - 2$ digits consisting of one 0 and one 1, that is $(k-1)^2$. Now $(k-1)^2 + (k-1) = k(k-1)$ is even, so the result follows by induction.

Noncanonical descriptions are allowed because the reduction mod 2 can convert a canonical description to a smaller noncanonical one, so it is important that the formulas are still valid for noncanonical descriptions.

Solution 4: Two-point induction using a close line segment

We work by induction on n. The result is clearly true for $n = 0$ and $n = 2$, so suppose that the result holds for all even $n < k$ for some even $k \geq 4$ and consider the case $n = k$.

We may clearly presume that P lies within the convex hull of $\mathcal{S}$ (otherwise it lies in no triangles). The straight line segments between points of $\mathcal{S}$ divide the convex hull into convex regions; suppose that one of the edges of the region containing P is from the line segment AB for some points A, $B \in \mathcal{S}$. By the induction hypothesis, the number of triangles containing P whose vertices are from $\mathcal{S} \setminus \{A, B\}$ is even, so it remains to show only that the number of triangles containing P which have at least one of A and B as a vertex is also even. The (infinite) line AB divides the points of $\mathcal{S} \setminus \{A, B\}$ into those on the same side as P, say p of them, and those on the other side, $n - 2 - p$ of them. P is in all triangles ABX where X is on the same side as P (since all such triangles contain the convex region containing P), and in no such triangles where X is on the other side. If X and Y are on the same side, P does not lie in AXY or BXY. If X and Y are on different sides, say X on the same side as P and Y on the other side, then since P lies in ABX it lies in the convex hull of $ABXY$, and so lies in exactly one of AXY and BXY by considering three cases (shown in Figure 1): the convex hull can be a quadrilateral (AXY and BXY disjoint and covering that quadrilateral), a triangle with B inside (AXY containing ABX which contains P and BXY being disjoint from ABX), or a triangle with A inside (BXY containing ABX which contains P and AXY being disjoint from ABX). Thus P lies in exactly $p + p(n - 2 - p) = p(n - 2) - p(p - 1)$ triangles with at least one of A and B as a vertex, $(n - 2)$ is even and so is one of p and $(p - 1)$.

This solution can also be expressed in terms of AB being the closest line segment to P, and I originally found it in that form. However, the notion of distance *does not appear in the original problem: it involves only notions of* ordered geometry*, as do the other five solutions. Thus expressing the solution without involving distance, as above, seems preferable.*

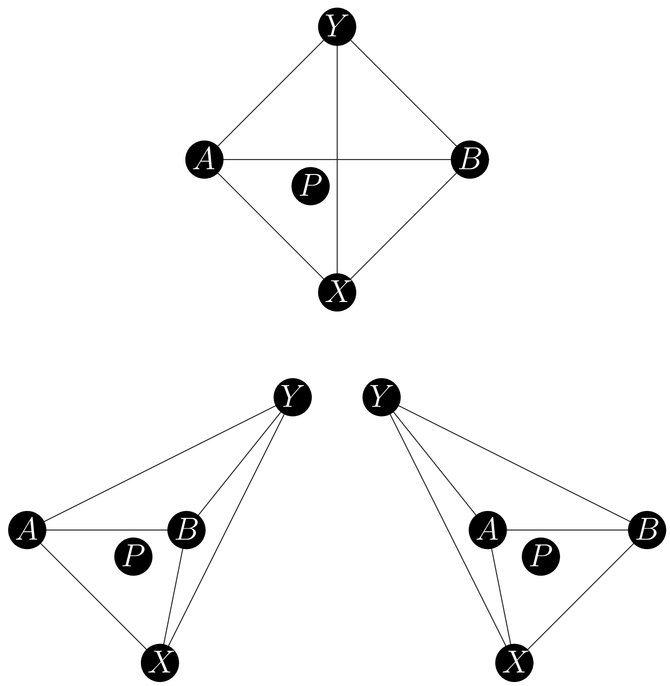

Figure 1: Cases for solution 4

Solution 5: Four-point induction

We work by induction on even integers n. We use the following induction hypothesis which is slightly stronger than the statement of the problem: for any set $\mathcal{S}$ of n points, no three collinear, and any point $P \notin \mathcal{S}$ not collinear with any two points of $\mathcal{S}$, the number of triangles containing P whose vertices are points of $\mathcal{S}$ is even, and for any two points A, $B \in \mathcal{S}$ the number of triangles containing P whose vertices are points of $\mathcal{S}$ and which have at least one of A and B as a vertex is even. This hypothesis is clearly true for $n = 0$ and $n = 2$, so suppose that the hypothesis holds for all even $n < k$ for some even $k \geq 4$ and consider the case $n = k$.

Choose any A, $B \in \mathcal{S}$. The number of triangles containing P whose vertices are points of $\mathcal{S} \setminus \{A, B\}$ is even by the induction hypothesis, so it only remains to consider triangles with A or B as a vertex. Choose some other C, $D \in \mathcal{S}$; by the induction hypothesis applied to $\mathcal{S} \setminus \{C, D\}$, the number of triangles containing P with at least one of A or B as a vertex but without C or D as a vertex is also even, So it now only remains to show that the number of triangles containing P, with at least one of A and B as a vertex, and with at least one of C and D as a vertex, is even.

First consider triangles all of whose vertices are from $\{A, B, C, D\}$. If P is outside the convex hull of $ABCD$ then it is in no such triangles, otherwise it is in two such triangles; the three essentially different cases are shown in Figure 2. (Note that there is symmetry between A and B; between C and D; and between the pair AB and the pair CD.)

Now consider triangles with some vertex X not from $\{A, B, C, D\}$. For each $X \in \mathcal{S} \setminus \{A, B, C, D\}$ we must count how many of the triangles ACX, ADX, BCX, BDX contain P. We shall show that (depending on the positions of P, A, B, C, D) this number is either even for all X or odd for all X; there being $n - 4$ such $X \in \mathcal{S} \setminus \{A, B, C, D\}$, the result will then follow. If we draw the lines PA and PC, we see that P lies in XAC if and only if X is in the sector of the plane bounded by the half-lines from P away from A and C. Thus we need to consider configurations of half-lines from P away from A, B, C, D (where the only property of configurations to consider is on which side of each pair of half-lines the angle is less than 180°). The six cases are shown in Figure 3; the lines away from A, B, C, D are marked A', B', C', D' and each sector is marked with the number of the triangles ACX, ADX, BCX, BDX containing P if X is in that sector.

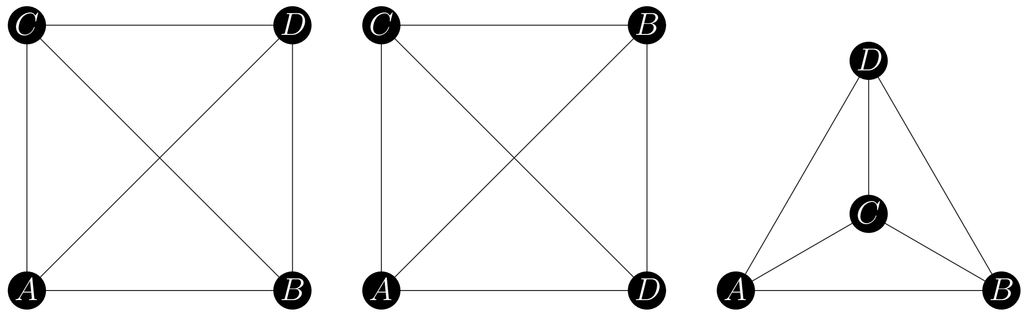

Figure 2: Cases for solution 5: triangles from $\{A, B, C, D\}$

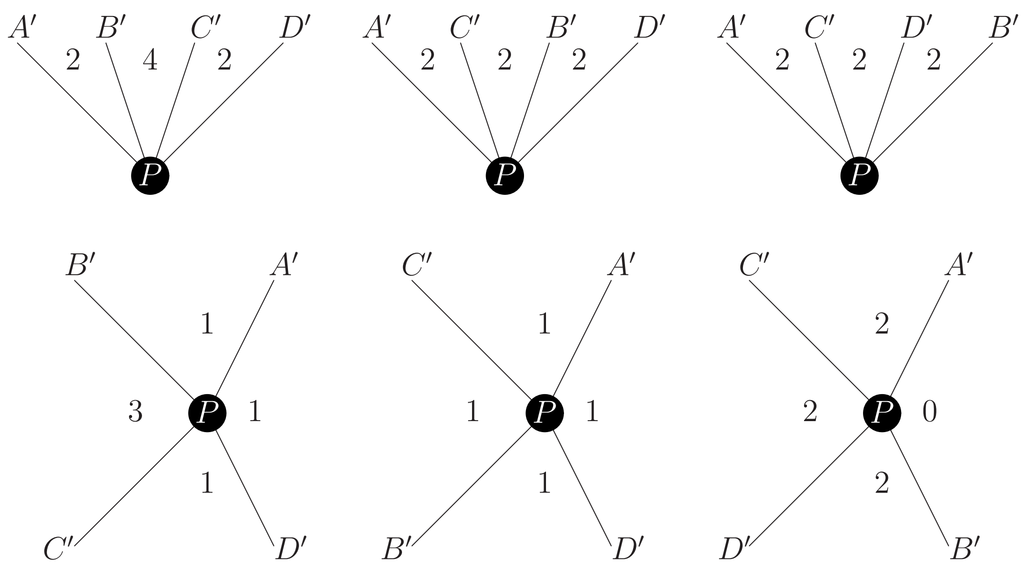

Figure 3: Cases for solution 5: plane sectors

Solution 6: Six-point induction

I do not recommend this solution, but it is the logical conclusion of applying the induction of the previous solution and not stopping at four points. As noted at the end, the analysis of cases could be avoided and replaced by one from the previous solution, but then it would just be a more convoluted version of that solution.

We work by induction on even integers n. We use the following induction hypothesis which is slightly stronger than the statement of the problem: for any set $\mathcal{S}$ of n points, no three collinear, and any point $P \notin \mathcal{S}$ not collinear with any two points of $\mathcal{S}$, the number of triangles containing P whose vertices are points of $\mathcal{S}$ is even, and for any two points A, $B \in \mathcal{S}$ the number of triangles containing P whose vertices are points of $\mathcal{S}$ and which have at least one of A and B as a vertex is even, and for any four points A, B, C, $D \in \mathcal{S}$ the number of triangles containing P whose vertices are points of $\mathcal{S}$ and which have at least one of A and B as a vertex and which have at least one of C and D as a vertex is even. This hypothesis is clearly true for $n = 0$ and $n = 2$, and holds for $n = 4$ by considering the cases of Figure 2 above, so suppose that the hypothesis holds for all even $n < k$ for some even $k \geq 6$ and consider the case $n = k$.

Choose any A, B, C, $D \in \mathcal{S}$. The number of triangles containing P whose vertices are points of $\mathcal{S} \setminus \{A, B\}$ is even by the induction hypothesis, as is the number whose vertices are points of $\mathcal{S} \setminus \{C, D\}$ but which have at least one of A and B as a vertex, so it only remains to consider triangles with A or B as a vertex and with C or D as a vertex. Choose some other E, $F \in \mathcal{S}$; by the induction hypothesis applied to $\mathcal{S} \setminus \{E, F\}$, the number of triangles containing P with at least one of A or B as a vertex and with at least one of C or D as a vertex but without E or F as a vertex is also even. So it now only remains to show that the number of triangles containing P, with at least one of A and B as a vertex, and with at least one of C and D as a vertex, and with at least one of E or F as a vertex, is even. But such triangles must have all their vertices from $\{A, B, C, D, E, F\}$, so we need only consider the possible configurations of six points and show that each region in each such configuration is contained in an even number of the eight triangles meeting the given conditions. Without loss of generality, we do not need to consider configurations where three lines between disjoint pairs of vertices are concurrent; slightly perturbing the vertices of such a configuration will yield one where all the regions of the previous configuration remain, in the same number of triangles as before, but a further region or regions have been added between the previously concurrent lines.

Considering essentially different configurations of points and line segments between them, we find one configuration of three points, two of four points, three of five points and sixteen of six points, as shown in Figure 4 (cases of fewer than six points), Figure 5 (cases of six points with more than three points on the convex hull) and Figure 6 (cases of six points with three points on the convex hull). Considering how the points of a configuration are paired (and allowing for the symmetry of some configurations), the configurations in Figure 5 and Figure 6 lead to respectively 7, 3, 9, 15, 11, 15, 15, 9, 11, 15, 5, 7, 9, 9, 15 and 15 cases, a total of 170 cases which may individually be checked if required to have each region in an even number of the eight triangles.

While the list of configurations is just presented as an assertion, such assertions of completeness and correctness should not necessarily be held as very reliable without a better-defined systematic means of enumeration of configurations, and so an enumeration differing from that listed here should not automatically be presumed to be incorrect.

It would be possible to avoid the above analysis of cases by considering the configuration of $ABCD$, then looking separately at E and F in relation to this configuration. However, each of E and F would then be in the position of X in solution 5 above, and so such a solution would be essentially a version of solution 5 but with a more complicated induction than necessary. To distinguish this solution from solution 5, we proceed to the full analysis of cases.

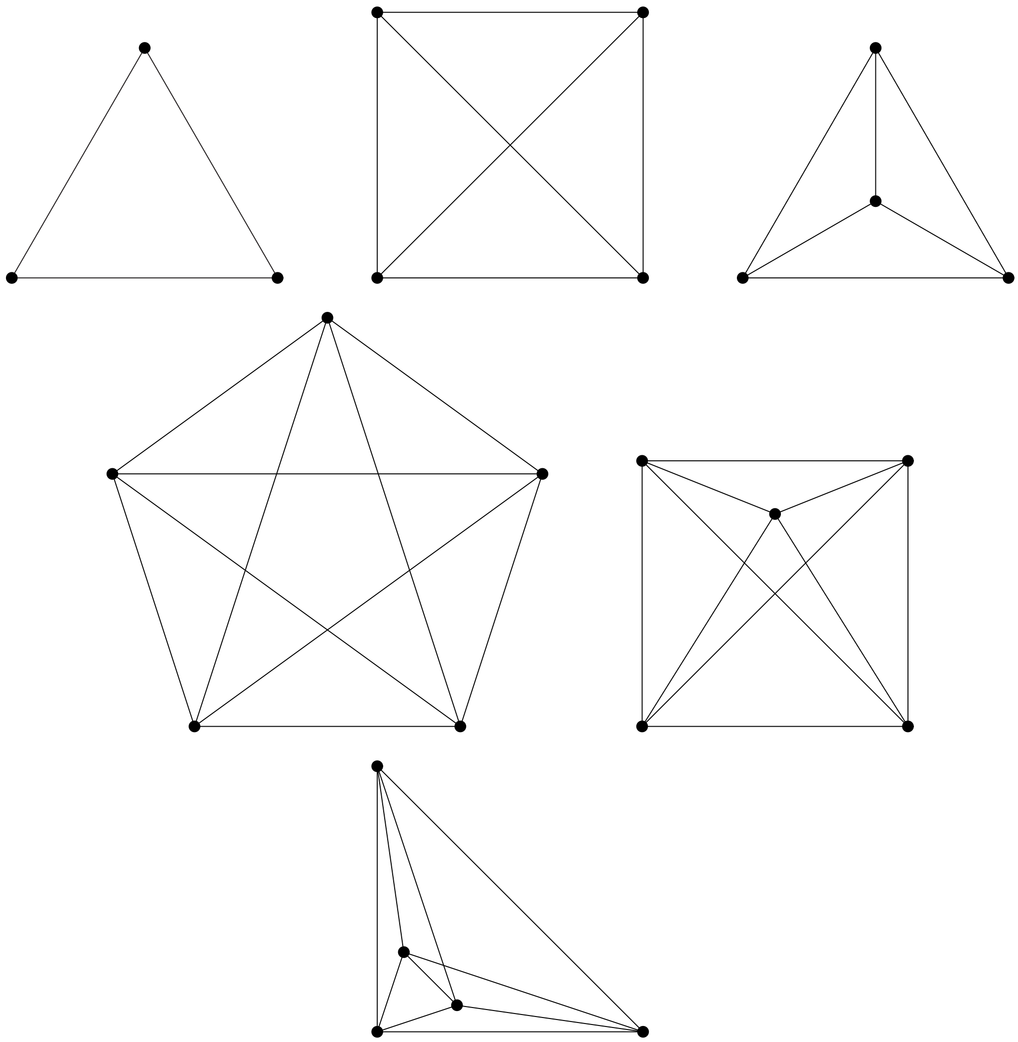

Figure 4: Cases for solution 6: three, four or five points

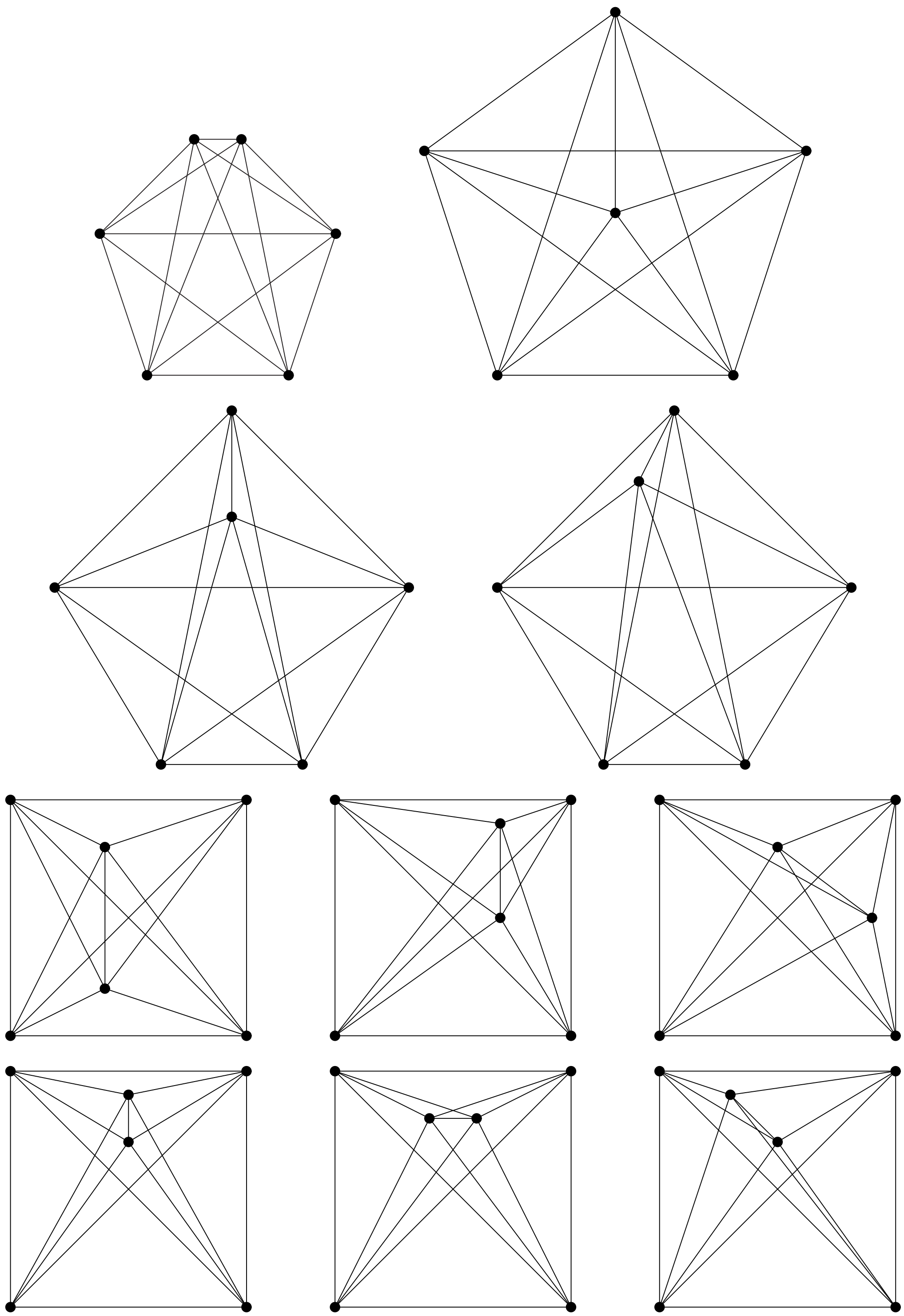

Figure 5: Cases for solution 6: six points, convex hull not a triangle

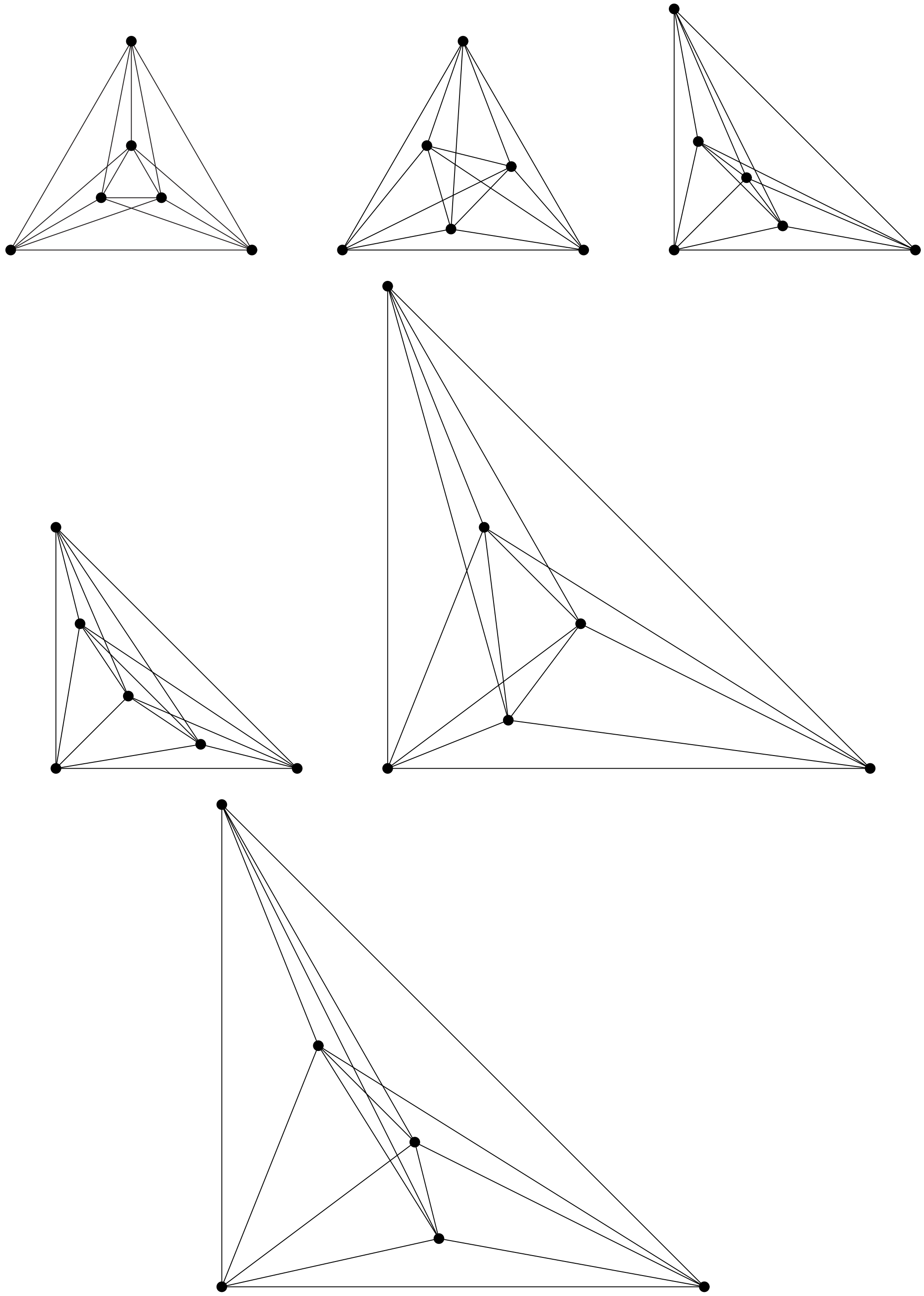

Figure 6: Cases for solution 6: six points, convex hull a triangle

Solution 7: Graph theory

Consider the triangles containing P (with vertices in $\mathcal{S}$) as vertices of a graph. Two triangles are joined by an edge in this graph if and only if they share two vertices in $\mathcal{S}$. Given a triangle containing P, and a fourth point of $\mathcal{S}$, P must be in exactly one triangle with vertices the new point and two vertices of the original triangle, by the properties of sets of four points discussed above under solution 2. Thus every vertex of this graph has degree $n-3$, which is odd, so the graph has an even number of vertices; that is, an even number of triangles contain P.

Solution 8: Rotating a line with binomial coefficients

Draw a straight line through P, not passing through any point of $\mathcal{S}$. Rotate this line clockwise around P; as it rotates is passes points of $\mathcal{S}$, until after having rotated by 180° it has passed every point exactly once (and continuing it passes each point each in the course of each 180° it turns).

$\binom{n}{3}$ is even, so the number of triangles containing P is even if and only if the number not containing P is even. We will count the number not containing P. Points X, Y, Z forming a triangle not containing P all lie on one side of the line at some point in its rotation; then one half line passes the points successively on some other, say X then Y then Z, after which again they all lie on one side (now the other side). In the course of rotation by 180°, X will be passed exactly once, so we arrange to count each triangle not containing P at the point where the line moves from having all points on one side to having one on one side and two on the other (passing X, in this case). Say that just before passing X there are k points on the same side as X and $n-k$ on the other; then there are $\binom{k-1}{2}$ triangles not containing P for which X is the first vertex in the ordering we determined. Thus, if we sum over all points X the value $\binom{k-1}{2}$ where k is the number of points on the same side as X just before the line passes X, the result is the number of triangles not containing P, which we wish to show is even.

Say we have r points on one side of the line and $n-r$ on the other side of the line. We keep track of the changing value of r. If this goes down, we have passed a point on that side, and add $\binom{r-1}{2}$ to the total number of triangles; if it goes up, we have passed a point on the other side, and add $\binom{n-r-1}{2}$ to the total number of triangles. After rotating through 180° the value r has become $n-r$.

Note that $\binom{s}{2}$ mod 2 depends only on s mod 4, and we see that $\binom{n-r-1}{2} \equiv \binom{r+n-2}{2}$, i.e., $\binom{r-2}{2}$ if $n \equiv 0 \pmod 4$ and $\binom{r}{2}$ if $n \equiv 2 \pmod 4$. If the number of points on one side goes down then up we have added $\binom{r-1}{2} + \binom{n-(r-1)-1}{2}$ to the number of triangles; this is odd for $n \equiv 0 \pmod 4$ and even for $n \equiv 2 \pmod 4$. If the number of points on one side goes up then down we have added $\binom{n-r-1}{2} + \binom{r}{2}$, again odd for $n \equiv 0 \pmod 4$ and even for $n \equiv 2 \pmod 4$.

By repeatedly cancelling pairs of successive rises and falls of r, we reduce to a straight path between r and $n-r$, of length $|n-2r|$, so having cancelled r such pairs (each yielding an odd number of triangles for $n \equiv 0 \pmod 4$, an even number for $n \equiv 2 \pmod 4$). At this point we could without loss of generality suppose the line started in a position such that $r = n/2$ (as it must pass through such a position between any other r and $n-r$), so

$n/2$ such pairs were cancelled, yielding an even number of triangles in any case. Alternatively, we may compute the necessary sum depending on the value of $r \pmod 4$ and on whether r is less than or greater than $n/2$.

Solution 9: Algebraic enumeration

Take an arbitrary line ℓ through P, not passing through any point of $\mathcal{S}$. Any triangle containing P has as vertices one point on one side of ℓ and two points on the other side. Suppose it contains A on one side and two points on the other side; the necessary and sufficient condition for two points B, C on the other side to be such that ABC contains P is that B and C lie on opposite side of the line AP. Suppose it contains A_1 and A_2 on one side and one point on the other side; the necessary and sufficient condition for B on the other side to be such that A_1A_2B contains P is that B lies in the sector of the other side bounded by the lines A_1P and A_2P.

Thus list the points on one side of the line, in angular order round P, as A_1, A_2, ..., A_k; let the lines A_iP divide the other side of ℓ into sectors S_0, S_1, ..., S_k (where A_iP lies between S_{i-1} and S_i), and let S_i contain a_i points of $\mathcal{S}$. The number of triangles containing P whose vertices are A_i and two points on the other side of ℓ is $(a_0+\cdots+a_{i-1})(a_i+\cdots+a_k)$, so the total number of triangles containing P with just one A_i as a vertex is $s_1 = \sum_{i<j}(j-i)a_ia_j$. The number of triangles containing P with vertices A_i and A_j (supposing $i < j$) is $(a_i+\cdots+a_{j-1})$, so the total number of triangles containing P with two A_i as vertices is $s_2 = \sum_i i(k-i)a_i$. We wish to show that $s_1 + s_2$ is even.

Now let $e = \sum_{i \text{ even}} a_i$ and $o = \sum_{i \text{ odd}} a_i$. Note that $e + o + k = n$, which is even. Working mod 2, we have $s_1 = \sum_{i<j,\, j-i \text{ odd}} a_ia_j = eo$. If k is odd then one of e and o is even so eo is even, and one of i and $(k-i)$ is even for all i, so s_2 is even and we are done. If k is even then, again mod 2, $s_2 = o$; e and o are both even or both odd and so $eo + o$ is even and again we are done.

Solution 10: Two-point induction by a result from the 1983 Kürschák Competition

We work by induction on n. The result is clearly true for $n = 0$ and $n = 2$, so suppose that the result holds for all even $n < k$ for some even $k \geq 4$ and consider the case $n = k$.

From the 1983 Kürschák Competition, problem 3, we know that there are points A and B in $\mathcal{S}$ such that P lies in no triangle ABC with $C \in \mathcal{S}$. By the induction hypothesis, the number of triangles containing P whose vertices are from $\mathcal{S} \setminus \{A, B\}$ is even, so it remains to show only that the number of triangles containing P which have at least one of A and B as a vertex (and so exactly one of them as a vertex) is also even. But if P lies in AXY then it must also lie in BXY by the results for 4-sets (since it does not lie in ABX or ABY), so it lies in an even number of triangles with A or B as a vertex.

Solution 11: General two-point induction

We work by induction on n. The result is clearly true for $n = 0$ and $n = 2$, so suppose that the result holds for all even $n < k$ for some even $k \geq 4$ and consider the case $n = k$.

Take *any* points A and B in $\mathcal{S}$. By the induction hypothesis, the number of triangles containing P whose vertices are from $\mathcal{S} \setminus \{A, B\}$ is even, so it remains to show only that the number of triangles containing P which have at least one of A and B as a vertex is also even. Suppose that there are p triangles ABX containing P (so that $n - 2 - p$ vertices Y have ABY not containing P). Then, using the results on 4-sets and denoting vertices as X if ABX contains P and as Y otherwise, triangles of the forms AX_1X_2 and BX_1X_2 do not contain P; for any X and Y exactly one of AXY and BXY contains P; and for any Y_1 and Y_2 an even number of AY_1Y_2 and BY_1Y_2 contain P. So the number of triangles containing P is, mod 2, equal to $p + p(n - 2 - p)$, which is even.

Solutions 4 and 10 can be considered special cases of this solution, but they were found first before this version was identified.

Solution 12: Algebraic enumeration, simple version

As in solution 9, take an arbitrary line ℓ through P, not passing through any point of $\mathcal{S}$. This time, however, if there are an even number of points on each side of ℓ then rotate it until it passes one point so that there are now an odd number on each side of ℓ. For any A on one side of ℓ, AP divides the other side of ℓ so that there are an even number of points of $\mathcal{S}$ on one side of AP and an odd number on the other side; so the number of triangles containing P with vertices A and two points on the other side is even. This applies for any A on either side of ℓ, so the result follows.

The algebra in solution 9 was attempted by a contestant, but without a convincing argument that the final result was even. However, contestants using this sort of approach all seemed to treat ℓ as 'horizontal' instead of arbitrary, and so missed the simplification to this solution.