Extremal Theory of Graph Minors and Directed Graphs

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The writing of this dissertation is my own original work. It takes due account of previous work as detailed in the Bibliography. The results, methods and proofs are original except insofar as they are indicated by references to the Bibliography to be derived from previous work, and are my own work, not performed in collaboration, except for those in Section 3.3 which are joint work with Andrew Thomason.

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Abstract

In the first part of this dissertation, the extremal theory of graph minors is developed as follows. The results of Bollobás, Catlin and Erdős showing how large a complete minor is found in a random graph are extended to showing how large a complete bipartite $K_{s,t}$ minor is found for given t, even up to $t = n - \log n$. The Hadwiger number of random graphs in a model where different parts of the graph have different edge probabilities is determined almost surely. For a class of dense graphs generalising that of complete bipartite graphs, 'blown-up' graphs, the extremal problem in terms of average degree is solved asymptotically, generalising results of Thomason, the extremal graphs being random graphs, and it is shown how a restricted class of blown-up graphs are 'critical' for this problem. For $K_{2,t}$ minors, the extremal problem is solved exactly (rather than asymptotically) with the exact best possible number of edges to force such a minor, the methods being substantially different from those for denser minors. For complete minors, it is shown that the extremal graphs are quasi-random in the sense of Chung, Graham and Wilson, or essentially disjoint unions of quasi-random graphs, answering a question of Sós. The extremal problem in terms of connectivity rather than average degree is also considered, with results that are significantly stronger than those in terms of average degree in the cases where they apply.

In the second part of this dissertation, extremal problems relating to directed graphs are considered. The minimum number of monotone subsequences of length $k + 1$ in a permutation of length n is considered; the extremal permutations are determined exactly for $k = 2$, and for $k > 2$ and $n \geq k(2k - 1)$ subject to an additional constraint, the number of extremal permutations being related to the Catalan numbers.

Introduction

Extremal graph theory, which essentially started with the work of Turán [68], concerns in its greatest generality the extremal values of some parameter in some class of graphs, and the nature of those graphs that attain the extremal values. A commonly considered parameter is that of the number of edges, or equivalently the average degree, and we may ask for the maximum average degree of graphs which do not contain some given substructure. Where the substructure is a complete subgraph, the exact extremal result was determined by Turán [68]; where it is some other nonbipartite subgraph, the extremal result is the Erdős-Stone-Simonovits theorem [16, 15].

In the first part of this dissertation, we develop the extremal theory of graph minors. The extremal problem for complete minors in terms of average degree was considered by Mader [36], and a succession of authors refined the bounds until the exact extremal result was found by Thomason [64]. Thomason provided an explicit form of the extremal graphs in terms of quasi-random [9] graphs, but the outline argument given to show that this is the extremal form is flawed. In Chapter 5 we answer a more general question of Sós about the form of graphs without large complete minors, and in so doing we correct and complete Thomason's argument.

In Chapter 3, we extend this extremal theory to cover a wider class of dense minors. The aim of this work is to find a structural property of H that determines the extremal function, analogous to the rôle of the chromatic number in the Erdős-Stone-Simonovits theorem; to some extent we succeed (for example, for dense regular graphs, and for almost all graphs), although without an exact characterisation for all cases; in joint work with Andrew Thomason, this is taken further in [41]. Because the extremal graphs are often random graphs, it is important to consider when given minors occur in random graphs, and we do this in Chapter 2, extending the work of Bollobás, Catlin and Erdős [2] on complete graphs as minors of random graphs.

Just as the extremal problems for bipartite subgraphs are less well understood than those for other subgraphs, the extremal problems for sparse minors are less well understood than those for dense minors. In Chapter 2 we consider in detail when complete bipartite graphs occur as minors of random graphs, and in this case we can obtain precise results that cover very sparse graphs as well as dense graphs. For dense minors, the graphs of maximum average degree that do not have a given minor turn out to be related to random graphs, but this does not happen for sparse minors. The specific case of $K_{s,t}$ minors, where s is fixed, is considered in Chapter 4, where a precise answer to the extremal problem is found for the case $s = 2$; there, provided t is sufficiently large, an average degree of $t+1$ forces a $K_{2,t}$ minor, but there are graphs with any smaller average degree and no $K_{2,t}$ minor.

Extremal problems for graph minors can, of course, be considered in terms of parameters other than the average degree. In Chapter 6, we consider the extremal problem for complete minors in terms of the connectivity, and obtain partial results that are better than those in terms of average degree in certain cases.

In the second part of this dissertation, we consider some other extremal problems relating to directed graphs. In Chapter 8, we consider briefly two long unsolved simple problems which seem to be related to each other, and give another conjecture which is also very simple and appears to be related to those problems. In Chapter 9, we consider an extremal problem related to the result on monotone subsequences of Erdős and Szekeres [17]. This problem, though not at first sight a problem relating to graphs, can be expressed as a problem on tournaments in a way which shows its natural symmetry, and this form of the problem allows a result in graph theory to be applied to show that certain sequences are extremal, and with a little more effort to give a complete characterisation of the extremal sequences for subsequences of length 3. A conjectured characterisation is also given for extremal subsequences of greater length, provided that the sequence in which subsequences are to be found is sufficiently long, and this is proved correct for a constrained version of the problem.

I would like to thank Andrew Thomason for his helpful comments and suggestions about the work that has gone into this dissertation. Much of Section 3.3 (dealing with the question of which graphs are critical when finding dense minors in both random and more general graphs) represents joint work with Andrew Thomason.

Notation

In general the terminology and notation of Bollobás [1] are used in this dissertation. Particular points to note are that graphs are simple and undirected unless stated otherwise; they are also finite throughout this dissertation. We write $H \prec G$ or $G \succ H$ to denote that H is a minor of G. We use $\mathbb{P}$ for probability and $\mathbb{E}$ for expectation. We write $\mathrm{Bi}(n,p)$ for the binomial distribution that is the sum of n independent random variables, each of which is 1 with probability p and 0 with probability $1-p$. We use the term *oriented graph* for an orientation of an undirected graph; in particular, an oriented graph may not have both edges $x \to y$ and $y \to x$. In a directed graph, $\Gamma^+(v)$ denotes the out-neighbourhood of v and $\Gamma^-(v)$ the in-neighbourhood. The most commonly used random graph model is that where each edge is independent and has the same probability of being present; where that probability is p, and the graph is of order n, we write this $\mathcal{G}(n,p)$. We write $\subset$ for subsets, where equality may occur.

Part I

Graph Minors

Chapter 1

Introduction to graph minors

We recall the standard definition of a minor:

Definition 1.1 *Let G be a graph. A* minor *or* subcontraction *of G is a graph H that can be obtained from G by a series of vertex and edge deletions and edge contractions. Equivalently, it is a graph H such that there exist disjoint subsets $W_u \subset V(G)$, for $u \in V(H)$, such that all $G[W_u]$ are connected and, for all $uv \in E(H)$, there is an edge in G between W_u and W_v. This relation is written $H \prec G$.*

Graphs which do not have some given minor, or which do not have any minor in some given set of graphs, have been studied and characterised in many ways. A major part of the theory of graph minors is in the series of papers by Robertson and Seymour [42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57], which aims towards proving Wagner's conjecture that, in any infinite set of finite graphs, some one is a minor of some other. In the course of this series of papers, various results are proved about the structure of graphs without a given minor, in terms of their concept of 'tree-width'. They also prove [49] a version of Kuratowski's theorem [33] for general surfaces, that for any surface there is a finite set of excluded minors, such that a graph can be drawn in that surface if and only if it does not have any of the excluded minors. (A simpler proof using the Robertson-Seymour theory has since been given by Thomassen [67]. The parts of the Robertson-Seymour theory used in this proof were further simplified by Diestel, Jensen, Gorbunov and Thomassen [13].)

Extremal problems in graph minors, concerning parameters other than tree-width, have also long been considered. Hadwiger [21] conjectured that $\chi(G) \geq k$ implies that $K_k \prec G$. The following definition is standard:

Definition 1.2 *The* Hadwiger number *of a graph G is the largest integer k such that $K_k \prec G$.*

Mader [36] showed that a sufficiently large average degree forces a K_t minor, which leads naturally to the question of what average degree is required. Bollobás, Catlin and Erdős [2] determined what order of complete minor occurs in a random graph, $n/\sqrt{\log_{1/q} n}$ for a $\mathcal{G}(n, 1-q)$ random graph; this showed Hadwiger's conjecture to hold for random graphs, and Fernandez de la Vega [18] observed that this showed that random graphs are good examples of graphs with high average degree but no large complete minor, and that it implied that the necessary average degree to force a K_t minor was not just a linear function of t. Kostochka [29, 30] showed that random graphs are within a constant factor of being optimal. The exact extremal function was then determined by Thomason [64]: the average degree that forces a K_t minor is $\big(1 + o(1)\big)\alpha t\sqrt{\log t}$ for an explicitly determined constant α. The random graphs achieving this extremum are graphs of a certain order and a fixed density λ.

Thomason [64] described the extremal graphs in terms of graphs that are quasi-random in the sense of Chung, Graham and Wilson [9] or Thomason [61], with an outline proof. That outline proof turns out to be flawed where it claims to show quasi-randomness, and I fill that gap in [39] and Chapter 5. Sós asked a more general question: sometimes quasi-random graphs contain larger minors than the corresponding random graphs (examples are given by Thomason [63]; and indeed the problem, raised by Mader, of explicitly presenting graphs without large complete minors remains open), but might the converse be true? That is, if a graph of order n and density p had no complete minor larger than that in a random graph $\mathcal{G}(n,p)$, would the graph then necessarily be quasi-random? This is answered in Chapter 5.

Bollobás, Catlin and Erdős [2] only give their proofs for graphs $\mathcal{G}(n, \frac{1}{2})$, although they note that the results may straightforwardly be extended to $\mathcal{G}(n,p)$ for any p; and their results are only for complete minors, as are those of Thomason [64], not for other minors H. In Chapter 2 and Chapter 3 we consider when more general graphs H occur as minors of random graphs. In particular, we find that large classes of graphs H (which include almost all graphs, and dense regular graphs) occur as minors in random graphs G just when complete graphs $K_{|H|}$ do. Graphs that are easier to find as minors in random graphs than complete graphs are must possess what we call a *tail*. In Chapter 3 we consider the corresponding extremal problem for more general dense graphs H (although not for all such H), showing the same relation to random graphs as was found by Thomason [64] for complete minors: the extremum is determined by random graphs of a certain order, and constant density λ. In joint work with Andrew Thomason [41], this is extended to wider classes of graphs, including some graphs that are sparse but not too sparse (having $t^{1+\tau}$ edges for positive τ).

Most of these results require H to be sufficiently large for its density, although in Chapter 2 we consider sparse complete bipartite minors of random graphs. One can also consider extremal problems for sparse minors H; in many such cases where H is very sparse, the extremal graphs are no longer random. We do not have a general theory for these graphs, but in Chapter 4 we consider the case where H is $K_{s,t}$ where s is fixed and t is large, and find a precise result when $s = 2$.

Extremal problems for graph minors can also be considered in terms of parameters other than the average degree. (Problems in terms of girth have been considered by Thomassen [66] and Kühn and Osthus [32]. Kühn [31] showed that large 'external connectivity' forces a large complete minor.) For given t, the extremal graphs (in terms of average degree) of large order with no K_t minor are made of dense quasi-random components joined by a few edges. If we require a certain connectivity, rather than a certain average degree, such graphs can no longer occur. Thus, we might expect that, for large n, a connectivity smaller than the $O\left(t\sqrt{\log t}\right)$ average degree would suffice to force a K_t minor. In Chapter 6 we consider this problem; in a specific case, we find that the necessary connectivity is linear in the order of the minor required. A stronger conjecture about the specific case of 6-connected graphs with no K_6 minor has been made by Jørgensen [28].

Chapter 2

Minors in random graphs

2.1 Introduction

Bollobás, Catlin and Erdős [2] considered the problem of determining the Hadwiger number of a $\mathcal{G}(n, \frac{1}{2})$ random graph, showing that it is almost surely $\big(1 + o(1)\big)n/\sqrt{\log_2 n}$. They did not state their results and proof for more general $\mathcal{G}(n, p)$ graphs, though they noted that they could straightforwardly be extended; for constant p, the Hadwiger number will almost surely be $\big(1+o(1)\big)n/\sqrt{\log_{1/q} n}$, for $q = 1-p$. (In fact they gave a more precise result for $p = \frac{1}{2}$, bounding the $o(1)$ term; in this chapter we only attempt to find results to within a $1 + o(1)$ factor.) The problem was also considered by McDiarmid [35].

It is natural to consider what other minors might be found in random graphs. After complete minors, the next simplest to consider are complete bipartite minors, which we consider in Section 2.2. If we fix the ratio $\beta : (1-\beta)$ of the parts of the minor, and ask for how large a t a $K_{\beta t,(1-\beta)t}$ minor can be found, it turns out that $t = \big(1+o(1)\big)n\Big/\sqrt{4\beta(1-\beta)\log_{1/q} n}$ almost surely; in particular, a $K_{t/2,t/2}$ minor is not significantly easier to find than a K_t minor.

Further, if $\beta \leq \frac{1}{2}$, it turns out that we can find a $K_{\beta t} + \overline{K_{(1-\beta)t}}$ minor just as easily as a $K_{\beta t,(1-\beta)t}$ one.

These questions can be extended in two directions. First, we can ask how large a complete bipartite minor can be found in a random graph if the ratio of the parts of the minor is not fixed, and one part is much larger than the other; that is, given t, we can ask for the largest s such that G has a $K_{s,t}$ minor. In Section 2.2 we see how an exact answer can be given to this, even up to $t = n - \log n$.

Second, we can look at more general minors than complete bipartite ones. We already saw how in some cases a larger minor is just as easy to find in a random graph as a smaller one; random graphs have $K_{t/2,t/2}$ minors essentially just when they have K_t minors. The notion of bipartite and multipartite graphs can be extended to a notion of *blown-up graphs*. These are discussed in Section 3.3 in the next chapter because the main use of the methods and results relating to these graphs is in considering questions of when a more general dense graph can be found as a minor, for which in many cases the extremal graphs turn out to be random.

All this work deals with the $\mathcal{G}(n, p)$ random graph model. Other random graph models can also be of interest. In Chapter 5 we are interested in graphs that are non-quasi-random; that is, where the density in different parts of the graph differs. In preparation for this, in Section 2.3 we consider random graphs where the edges are independent but the edge probabilities vary; given a fixed vertex partition (X, Y), the edge probabilities within X, within Y and between X and Y differ (but are fixed within each of these three parts). We determine the Hadwiger number of such graphs (almost surely).

2.2 Complete bipartite minors

In this section, we fix some edge probability p, with $0 < p < 1$, write as usual $q = 1 - p$, and consider the question: for given t (which may be a function of n), what is the largest s such that $K_{s,t} \prec \mathcal{G}(n,p)$? We also consider the related question: what is the largest s for which $K_s + \overline{K_t} \prec \mathcal{G}(n,p)$? For both of these questions, we find the value that s holds almost surely, to within a $o(1)$ term. Where $t \geq n/2\sqrt{\log_{1/q} n}$, the answers to both questions are (to within this $o(1)$ term) the same.

For a graph G, and a positive integer t, define $s_t(G)$ to be the largest positive integer s such that $K_{t,s}$ is a minor of G, with $s_t(G) = 0$ if there is no such minor for any positive s. Define $s_t^+(G)$ similarly where we require the minor to be $K_s + \overline{K_t}$.

For positive integers n, t and real $0 < q < 1$, put

$$\ell_n(t) = \left\lfloor \frac{n/t + 1}{2} \right\rfloor,$$

and put

$$s_{n,q}(t) = \frac{\ell_n(t)\big(n - \ell_n(t)t\big)}{\log_{1/q} n}.$$

It will turn out that $\ell_n(t)$ is the optimal order of the parts of the minor on the t-side, with those on the s-side being of order $(\log_{1/q} n)/\ell_n(t)$. The value of $\ell_n(t)$ arises from maximising $(n - \ell t)\ell$ for integer ℓ. For large t (and so small s) this means that there are points at which the distribution of vertices of G between the parts of the minor corresponding to each half of the bipartite graph jumps; the number of vertices in each part of the t-side of the minor goes up, and the number in each part of the s-side goes down. If $t = o(n)$, however, the results can be expressed more simply, as the integral parts need no longer be taken; if $\omega > 1$ and we have $t = \omega n/2\sqrt{\log_{1/q} n}$ then $s = n/2\omega\sqrt{\log_{1/q} n}$.

In general, we claim that, for fixed $q = 1 - p$, $s_t\big(\mathcal{G}(n,p)\big) = \big(s_{n,q}(t) + O(1)\big)\big(1 + o(1)\big) = s_t^+\big(\mathcal{G}(n,p)\big)$ almost surely as n tends to infinity, for any $t = t_n$ such that $n/2\sqrt{\log_{1/q} n} \le t_n < n$. The $O(1)$ term is $+1$ when we are showing that larger minors are not present, and -1 when we are showing that minors of the required size are present; it is only relevant when t is very near n and there is uncertainty as to the exact number of parts that can be found on the s-side. The precise meaning of the claim is given in the following theorems. Theorem 2.1 says that the minor is no larger than claimed; Theorem 2.2 says that the minor is no smaller.

Theorem 2.1 *Let $0 < p < 1$ be fixed, and put $q = 1 - p$. Let $0 < \epsilon < \frac{1}{5}$ be given. Let t_n be such that $n/2\sqrt{\log_{1/q} n} \le t_n < n$ for all n. Put $s_n = \big\lceil(1+\epsilon)\big(s_{n,q}(t_n) + 1\big)\big\rceil$. Then, for all sufficiently large n, a random graph $\mathcal{G}(n,p)$ contains a K_{t_n,s_n} minor with probability at most ϵ.*

Proof If $G = \mathcal{G}(n,p)$ has such a minor, the vertices may be partitioned into t_n sets $T_1, \ldots, T_{t_n}$ and s_n sets $S_1, \ldots, S_{s_n}$ with an edge between each T_i and each S_j. We say a partition of the vertices of G into $T_1, \ldots, T_{t_n}$, $S_1, \ldots, S_{s_n}$ is *permissible* if it has such edges between T_i and S_j for all $1 \le i \le t_n$, $1 \le j \le s_n$. For G to have such a minor, it must have a permissible partition; note that there are at most n^n possible partitions.

Let the probability that a given partition (of the fixed vertex set of order n, while G is random) is permissible be P. We then have

$$P = \prod_{i,j}\left(1 - q^{|T_i||S_j|}\right) \le \exp\left(-\sum_{i,j} q^{|T_i||S_j|}\right).$$

The right hand side of this inequality is maximised when the sum is minimised. The sum will be minimised for some choice of the $|T_i|$ and $|S_j|$ adding to n; considering holding all but $|T_{i_1}|$ and $|T_{i_2}|$ fixed, and differentiating, we see that the sum is minimised when all the $|T_i|$ are as nearly equal to each other as possible, and similarly all the $|S_j|$ are as nearly equal to each other as possible.

Applying the AM-GM inequality, we conclude the minimum to be when $\sum |T_i| = \sum |S_i| = n/2$, though this will not be achieved in certain cases because the sets must have integer sizes. We shall now consider three cases, in the first of which the approximation given by AM-GM shall suffice.

Case 1: Let $\omega(n) > 1$ be such that $\omega(n) \to \infty$ as $n \to \infty$. Consider those t_n with $t_n < n/\omega(n)$. We then have $\ell_n(t_n) = n/2t_n + o(1)$, and so

$$s_{n,q}(t_n) = \big(1 + o(1)\big)(n^2/4t_n \log_{1/q} n) > \big(1 + o(1)\big)\left(n\omega(n)/4\log_{1/q} n\right);$$

so

$$s_n = \big(1 + o(1)\big)(1 + \epsilon)s_{n,q}(t_n).$$

We then have $s_n t_n = \big(1 + o(1)\big)(1 + \epsilon)(n^2/4\log_{1/q} n)$.

A given partition is permissible with probability at most

$$\begin{aligned} &\exp\left[-t_n s_n q^{n^2/4t_n s_n}\right] \\ &\qquad = \exp\left[\left(-(1+\epsilon)n^2/4\log_{1/q} n\right) n^{-1/(1+\epsilon)(1+o(1))}\big(1+o(1)\big)\right] \\ &\qquad = \exp\left[-n^{2-1/(1+\epsilon)+o(1)}(1+\epsilon)\big(1+o(1)\big)/4\log_{1/q} n\right]. \end{aligned}$$

There are at most n^n partitions, so the probability that any one of them is permissible is at most

$$\exp\left[n\log n - \big(1 + o(1)\big)(1 + \epsilon)n^{2-1/(1+\epsilon)+o(1)}/4\log_{1/q} n\right]$$

which tends to zero as $n \to \infty$, so is less than ϵ for n sufficiently large. Thus the result holds in Case 1.

Case 2: Now consider those $t_n \geq n/\frac{1}{2}\log\log n$. We have $1 \leq \ell_n(t_n) < \log\log n$. Suppose also that $t_n < n(1-\epsilon^2)$. Suppose that the $|T_i|$ and $|S_j|$ are chosen to maximise the probability that a partition is permissible. Observe that by our choice of t_n, we have $s_n > \epsilon^2 n/\log_{1/q} n$.

Suppose the t_n sets T_i contain a total of rn vertices. Let those sets have orders m and $m+1$, where $m+1 < \log\log n$; if all T_i have the same order, let that order be m. Say un/m of the sets have order m, and $vn/(m+1)$ of the sets have order $m+1$, where $u+v=r$. Put $(1-r)n/s_n = w\log_{1/q} n$.

The probability that a given T_i has edges to all S_j is

$$\begin{aligned}\prod_j \left(1 - q^{|T_i||S_j|}\right) &\leq \exp\left[-\sum_j q^{|T_i||S_j|}\right] \\ &\leq \exp\left[-s_n q^{|T_i| w \log_{1/q} n}\right] \\ &= \exp\left[-s_n n^{-w|T_i|}\right]\end{aligned}$$

using AM-GM. The probability that any partition is permissible is thus at most

$$\exp\left[n\log n - s_n\left((u/m)n^{1-wm} + \big(v/(m+1)\big)n^{1-w(m+1)}\right)\right].$$

Suppose this probability is at least ϵ for some arbitrarily large n; then in particular we have, for all $\delta > 0$,

$$s_n\left((u/m)n^{1-wm} + \big(v/(m+1)\big)n^{1-w(m+1)}\right) < n^{1+\delta}$$

for some arbitrarily large n; whence, given our lower bound on s_n,

$$\left((u/m)n^{1-wm} + \big(v/(m+1)\big)n^{1-w(m+1)}\right) < n^{\delta}$$

again for arbitrarily large n. (The value $\delta = \epsilon^4$ will be sufficiently small for what follows.)

If $w \leq \big(1/(m+1)\big)(1-\epsilon^2)$, this cannot hold; so $1/w < (1+2\epsilon^2)(m+1)$. Also, if $u/m > 1/\log n$ (say), then we must also have $w > (1/m)(1-\epsilon^2)$;

so $1/w < (1+2\epsilon^2)m$. Clearly we have $m \leq \lfloor rn/t_n \rfloor$. If $u/m \leq 1/\log n$, then we have $rn \geq n/\log n + (t_n - n/\log n)(m+1)$, so $rn/t_n \geq (m+1) - mn/t_n \log n$. Thus, for arbitrarily large n, in either case we have $1/w < (1+2\epsilon^2)\lfloor rn/t_n + \epsilon^4 \rfloor$. Thus $s_n < \big((1-r)n/\log_{1/q} n\big)(1+2\epsilon^2)\lfloor rn/t_n + \epsilon^4 \rfloor$.

Put $\ell = \lfloor rn/t_n + \epsilon^4 \rfloor$; so $rn \geq \ell t_n - \epsilon^4 t_n \geq \ell t_n - \epsilon^4 n$. Thus we have arbitrarily large n with $s_n < n(1+\epsilon^4 - \ell t_n/n)(1+2\epsilon^2)\ell/\log_{1/q} n$. The right hand side of this is maximal, varying ℓ over the reals, for $\ell = (1+\epsilon^4)n/2t_n$. For ℓ an integer, this is maximal at the nearest integer value, and we have $1+\epsilon^4 - \ell t_n/n \leq (1+\epsilon^2)(1-\ell t_n/n)$ (the extremal case being when $t_n = (1-\epsilon^2)n$). Thus, $s_n < (1+5\epsilon^2)n(1-\ell t_n/n)\ell/\log_{1/q} n$; and for integer ℓ this is maximised for $\ell = \ell_n(t_n)$. But since $\epsilon < \frac{1}{5}$, we have $1+5\epsilon^2 < 1+\epsilon$, contradicting our original choice of s_n.

Case 3: Finally, suppose $t_n > n(1-\epsilon^2)$. Say $n - t_n = \beta n$, where $\beta < \epsilon^2$. Thus at least $n(1-2\beta)$ of the T_i are of order 1. If some S_i has order $w \log_{1/q} n$, the probability a given T_i of order 1 has an edge to it is $1 - n^{-w} < \exp(-n^{-w})$; so all T_i have edges to it with probability at most $\exp\big(-(1-2\beta)n^{1-w}\big)$. We are given that

$$s_n \geq (1+\epsilon)\left(1 + (\beta n/\log_{1/q} n)\right).$$

If r of the S_i have order at most $(1-\epsilon/2)\log_{1/q} n$, then

$$(s_n - r)(1-\epsilon/2)\log_{1/q} n \leq \beta n,$$

so

$$(s_n - r) \leq \beta n/(1-\epsilon/2)\log_{1/q} n$$

whence

$$\begin{aligned} r &\geq s_n - \beta n/(1-\epsilon/2)\log_{1/q} n \\ &> 1 + (\epsilon/3)\beta n/\log_{1/q} n. \end{aligned}$$

Thus, the probability that a given partition is permissible is at most

$$\exp\left[-(\epsilon/3)\beta(1-2\beta)n^{1+\epsilon/2}/\log_{1/q} n - (1-2\beta)n^{\epsilon/2}\right].$$

There are at most $n^{4\beta n}$ possible partitions in which at least $(1-2\beta)n$ parts are of order 1. Thus the probability that any partition is permissible is at most

$$\exp\left[4\beta n \log n - (\epsilon/3)\beta(1-2\beta)n^{1+\epsilon/2}/\log_{1/q} n - (1-2\beta)n^{\epsilon/2}\right].$$

For n sufficiently large this is less than ϵ (independently of β). □

Theorem 2.2 *Let* $0 < p < 1$ *be fixed,* $q = 1-p$. *Let* $0 < \epsilon < \frac{1}{2}$ *be given. Let* t_n *be such that* $n/2\sqrt{\log_{1/q} n} \leq t_n < n$ *for all* n. *Put* $s_n = \lfloor (1-\epsilon)\big(s_{n,q}(t_n) - 1\big)\rfloor$. *Then, for all sufficiently large* n, *a random graph* $\mathcal{G}(n,p)$ *contains a* $K_{s_n} + \overline{K_{t_n}}$ *minor with probability at least* $1-\epsilon$.

Proof Fix n and put $\ell = \ell_n(t_n)$. Put $k = \lceil (1+\epsilon)(\log_{1/q} n)/\ell \rceil$. Then there are enough vertices in the graph to fit t_n components of order ℓ and $(1+\epsilon^2)s_n$ components of order k; we claim that these components can be chosen to be connected, and, after a few of the components of order k are removed if necessary, leaving at least s_n of them, with an edge between each component of order ℓ and each component of order k, and an edge between each pair of components of order k (so we have our minor) with probability at least $1-\epsilon$.

Choose first at random the $t_n\ell$ vertices of one side; our T_i shall be chosen from these vertices, and the S_j from the remaining vertices. We shall find, within each side, our connected subgraphs, considering only the vertices of that side. When that is done, the probability that a given pair (S_i, T_j) has an edge between them is $1 - q^{k\ell} \geq 1 - q^{(1+\epsilon)\log_{1/q} n} = 1 - n^{-1-\epsilon}$. Say a pair is *bad* if it has no such edge. Thus the expected number of bad pairs is at most $t_n(1+\epsilon^2)s_n n^{-1-\epsilon} \leq (1+\epsilon^2)s_n n^{-\epsilon}$, so the probability that more than $\epsilon^2 s_n/2$ pairs are bad does not exceed $2(1+\epsilon^{-2})n^{-\epsilon}$ which is less than $\epsilon/6$ for n sufficiently large in terms of ϵ. Similarly, say that a pair (S_i, S_j), with distinguished vertices $u \in S_i$ and $v \in S_j$, is *bad* if there is no edge between $S_i - u$ and $S_j - v$; such a pair has an edge between them with probability $1 - q^{(k-1)^2} \geq 1 - q^{k\ell}$ (since $k > (1+\epsilon)\ell$ by the lower bound on t_n), so again the probability that more than $\epsilon^2 s_n/2$ pairs are bad is less than $\epsilon/6$ for n sufficiently large.

It now remains to find our connected subgraphs (with, for each side, a probability of at most $\epsilon/3$ of failing to find them on that side). We need to consider two cases.

First, suppose $t_n > n/3$, so that $\ell = 1$. The subgraphs of order 1 are trivially connected. The probability that one of the subgraphs of order k (chosen at random) is not connected is approximately the probability that it has an isolated vertex, which is not more than $(1+\epsilon)(\log_{1/q} n)n^{-1-\epsilon}$; so the probability than any of these subgraphs is disconnected is no more than $n^{-\epsilon}$ which is less than $\epsilon/3$ for n large.

Now suppose $t_n \leq n/3$. We wish to find connected subgraphs from each of the two chosen sets of vertices. We have enough room for one side to have $(1+\epsilon^4/4)t_n\ell$ vertices and the other to have $(1+\epsilon^2)s_n k$ vertices.

Thus, suppose we have $\ell m(1+\delta)$ vertices, where $2 \leq \ell \leq \log n$ is the order of the components we wish to find, $\delta = \epsilon^4/4$, and $m \geq 2n/3\log_{1/q} n$ is the number of such components we wish to find. It will suffice for the probability that such components can be found to be at least $1 - \epsilon/3$.

Let the set of vertices be X, and let T be a randomly chosen set of m vertices of X. Put $W = X \setminus T$. We wish to find m vertex-disjoint stars of order ℓ in X, with centres the vertices of T. If X is the set of vertices from which we choose our S_i, then we make T be the distinguished vertices in the definition of bad pairs above; and finding the stars is independent of the edges within $X \setminus T$, so that the determination of probabilities of bad pairs is valid.

If there are no such stars, then by a trivial corollary of Hall's theorem [22] there is a set $A \subset T$ with less than $(\ell-1)|A|$ neighbours in W. Write $|A| = a$. The probability that a given vertex in W is joined to no vertex of A is q^a, so the probability that A has less than $(\ell-1)a$ neighbours in W is at most

$$\sum_{u<(\ell-1)a} \binom{m\big(\ell(1+\delta)-1\big)}{u} q^{a(m(\ell(1+\delta)-1)-u)}$$

$$\begin{aligned}
&\leq (\ell-1)an^{(\ell-1)a}q^{a(m(\ell(1+\delta)-1)-(\ell-1)a)} \\
&\leq n^{\ell a}q^{am((\ell(1+\delta)-1)-(\ell-1))} \\
&\leq n^{\ell a}q^{am\ell\delta} \\
&= \big(n^{\ell}q^{m\ell\delta}\big)^a \\
&= \exp\left[a(\ell \log n + m\ell\delta \log q)\right] \\
&\leq \exp\left[a\ell\left(\log n + (2n/3\log_{1/q} n)\delta \log q\right)\right].
\end{aligned}$$

There are at most m^a sets A of the given size, so we see that the probability that any A has the given property is less than $\epsilon/3$ for n sufficiently large. □

Given $t \geq n/2\sqrt{\log_{1/q} n}$, we have now seen how large a $K_{s,t}$ or $K_s + \overline{K_t}$ minor a random graph $\mathcal{G}(n,p)$ has. It remains to be seen how large a $K_t + \overline{K_s}$ minor it has, and how large the minors are if we have $t < n/2\sqrt{\log_{1/q} n}$ (and so $s > t$). For the former, the minor has essentially no more vertices than a complete minor has, since $K_{(t+s)/2,(t+s)/2}$ is a subgraph of $K_t + \overline{K_s}$ if $s < t$.

For the latter, the results are corollaries of those where $t \geq n/2\sqrt{\log_{1/q} n}$

and the result of Bollobás, Catlin and Erdős generalised to $p \neq \frac{1}{2}$. It might have seemed more natural in the first place to consider the question in terms of being given $s < n/2\sqrt{\log_{1/q} n}$ and being asked for t. However, inverting the formulae for $\ell_n(t)$ and $s_{n,q}(t)$ given above, it is not hard to see that we arrive at the following, for positive integers n, s and real $0 < q < 1$:

$$\ell'_{n,q}(s) = \left\lfloor \frac{1 + 2(\frac{s}{n}) \log_{1/q} n + \sqrt{1 + 4(\frac{s}{n})^2 (\log_{1/q} n)^2}}{2} \right\rfloor,$$

and

$$t_{n,q}(s) = \frac{n - s(\log_{1/q} n)/\ell'_{n,q}(s)}{\ell'_{n,q}(s)}.$$

Here $\ell'_{n,q}(s)$ is the number of vertices in each part of the t-side of the minor; each part of the s-side has $(\log_{1/q} n)/\ell'_{n,q}(s)$ vertices. The largest t_n for which a random graph $\mathcal{G}(n, p)$ (with $q = 1 - p$) has a K_{s_n,t_n} or $K_{s_n} + \overline{K_{t_n}}$ minor is then almost surely $\big(1+o(1)\big)t_{n,q}(s_n)$. The substantially increased complexity of the formulae indicates that it was appropriate to consider the problem first in terms of given t, deriving this case as a corollary.

2.3 Constrained random graphs

In this section only, we consider a different random graph model, where edges are independent but different edges have different probabilities of being present; there is a fixed vertex partition (X, Y), and the edge probabilities within X, within Y and between X and Y differ (but are fixed within each of these three parts). This is closely related to graphs where the density of different parts of the graph is different; such graphs are considered in general in Chapter 5. Here we determine an upper bound on the order of complete minors in such random graphs; the corresponding lower bound, and so the determination of the Hadwiger number of these graphs to within a factor of

$1+o(1)$, is done for more general graphs in Theorem 5.2, and random graphs almost surely satisfy the connectivity condition of that theorem.

Theorem 2.3 *Let* $0 < \epsilon, \alpha, p_X, p_Y, p_{XY} < 1$ *be given. Put* $q_X = 1 - p_X$, $q_Y = 1 - p_Y$ *and* $q_{XY} = 1 - p_{XY}$. *Put*

$$q_* = q_X^{\alpha^2} q_Y^{(1-\alpha)^2} q_{XY}^{2\alpha(1-\alpha)}.$$

For all sufficiently large n, *if* $t = \left\lceil (1+\epsilon)n/\sqrt{\log_{1/q_*} n}\,\right\rceil$, *and* G *is a random graph on* n *vertices with a fixed partition of the vertices into* X *and* Y, *with* $|X| = \alpha|G|$, *where edges within* X *are present with probability* p_X, *edges within* Y *are present with probability* p_Y, *and edges between* X *and* Y *are present with probability* p_{XY}, *then* $\mathbb{P}(K_t \prec G) < \epsilon$.

Proof If G has such a minor, it has a vertex partition into parts $X_i \cup Y_i$, for $1 \le i \le t$, where $X_i \subset X$ and $Y_i \subset Y$, with an edge between any $X_i \cup Y_i$ and any other $X_j \cup Y_j$. There are at most n^n such partitions, and the probability that any given partition has the required edges is

$$\begin{aligned}
P &= \prod_{i \ne j} \left(1 - q_X^{|X_i||X_j|} q_Y^{|Y_i||Y_j|} q_{XY}^{|X_i||Y_j|+|X_j||Y_i|}\right) \\
&\le \exp\left[-\sum_{i \ne j} q_X^{|X_i||X_j|} q_Y^{|Y_i||Y_j|} q_{XY}^{|X_i||Y_j|+|X_j||Y_i|}\right] \\
&\le \exp\left[t/2 - \frac{1}{2}\sum_i \sum_j q_X^{|X_i||X_j|} q_Y^{|Y_i||Y_j|} q_{XY}^{|X_i||Y_j|+|X_j||Y_i|}\right].
\end{aligned}$$

The right hand side of this inequality is maximised when the sum is minimised. The product of the terms summed does not depend on the $|X_i|$ and $|Y_j|$, so the sum is minimised when all the terms are equal, which occurs when all $|X_i| = |X|/t = \alpha\sqrt{\log_{1/q_*} n}/(1+\epsilon)$ and all $|Y_i| = |Y|/t = (1-\alpha)\sqrt{\log_{1/q_*} n}/(1+\epsilon)$. We then have $|X_i||X_j| = \alpha^2(\log_{1/q_*} n)/(1+\epsilon)^2$,

$|Y_i||Y_j| = (1-\alpha)^2(\log_{1/q_*} n)/(1+\epsilon)^2$, and $|X_i||Y_j| + |X_j||Y_i| = 2\alpha(1-\alpha)(\log_{1/q_*} n)/(1+\epsilon)^2$, so

$$\begin{aligned} P &\leq \exp\left[t/2 - \tfrac{1}{2}t^2 q_*^{(\log_{1/q_*} n)/(1+\epsilon)^2}\right] \\ &= \exp\left[t/2 - \tfrac{1}{2}t^2 q_*^{(\log_{1/q_*} n)/(1+\epsilon)^2}\right] \\ &= \exp\left[t/2 - \tfrac{1}{2}t^2 n^{-1/(1+\epsilon)^2}\right]. \end{aligned}$$

Thus the probability that any partition has the required edges is less than ϵ.

□

Chapter 3

Noncomplete minors

3.1 Introduction

Thomason [64], following previous work by Mader [36], Kostochka [29, 30], and others, determined the extremal function for complete minors in terms of the average degree, showing that the average degree that forces a K_t minor is that of random graphs of a certain order and density, in which a K_t minor is almost surely the largest complete minor by the results of Bollobás, Catlin and Erdős [2]. If we define

$$c(t) = \min\{\, c : e(G) \geq c|G| \text{ implies } K_t \prec G \,\}$$

then $c(t)$ exists and he showed that $c(t) = \bigl(\alpha + o(1)\bigr)t\sqrt{\log t}$, where $\alpha = 0.3190863431\ldots$ is an explicit constant; or, equivalently, that the minimum average degree guaranteeing a K_t minor is $\bigl(2\alpha + o(1)\bigr)t\sqrt{\log t}$. This is the same Hadwiger number as for random graphs of density $p = 1 - \lambda$, where $\lambda = 0.2846681370\ldots$ is another explicit constant, and order $n = t\sqrt{\log_{1/\lambda} t}$.

It is natural to ask, for more general noncomplete but dense graphs, what average degree forces them as a minor; it will turn out that for the class we will define of 'blown-up graphs' this also derives from random graphs of the same density. We define

$$c(H) = \inf\{\, c : e(G) \geq c|G| \text{ implies } G \succ H \,\}.$$

Ideally, we would like an answer in the form of a structural property of H that determines the approximate value of $c(H)$, analogous to the rôle of the chromatic number in the Erdős-Stone-Simonovits theorem [16, 15] determining how many edges force an H subgraph. Failing this, we would like to determine $c(H)$ for as wide a class of H as possible.

Let F be a graph with no vertices of degree zero, where loops but not multiple edges are allowed. Let positive weights $w(v)$, summing to 1, be assigned to the vertices of F. Then $(F, \mathbf{w})$ can be blown up to a graph F_t of order t by mapping each vertex v to a set $W(v)$ of vertices with $\lfloor w(v)t \rfloor \leq |W(v)| \leq \lceil w(v)t \rceil$, where there is an edge between $u_1 \in W(v)$ and $u_2 \in W(w)$ (with $u_1 \neq u_2$, but possibly $v = w$) if and only if there is an edge between v and w in F. Note that, though F may have loops, F_t does not. (The graph F_t is not quite unique, since the exact size of each vertex set $W(v)$ is not precisely specified.) There is a natural sense in which certain $(F, \mathbf{w})$ are *critical* for blown-up minors; this is made precise in Definition 3.2. We shall see that this notion of criticality does not depend on the edge density of the graphs in which the minors are to be found, but only on $(F, \mathbf{w})$ itself, and in Section 3.3 shall find an exact characterisation of the critical graphs. We shall also see that any dense graph H is (apart from a few edges) a subgraph of a blow-up of a much smaller critical weighted graph which is no harder than H to find in a random graph or in a general dense graph. This work, in Section 3.3, is joint work with Andrew Thomason.

Thus, it would suffice to give a solution to the extremal problem for graphs taking the form of a blow-up plus a few edges. Unfortunately, for some dense graphs it is the sparse set of edges that determines when the dense graph can be found as a minor. We do not have a complete theory for such cases, although in joint work with Andrew Thomason [41] we show that the extremal graphs are random graphs (albeit without finding the desired structural property), but can give complete results for blown-up graphs. For this the methods of [64] can be adapted. Just as for minors in random graphs, the key feature is arranging for the different parts of the minor to have edges between them, each part then being made connected using a few spare vertices. Here it turns out that the method of equipartitions in [64] can be adapted to the task of finding blown-up minors; the key adaptation is assigning, at random, vertices of G that correspond to each vertex of F (the graph being blown up), before using the methods of [64] to assign, in a suitably constrained way, the vertices corresponding to each vertex of the blown-up graph F_t.

The main result for blown-up graphs is Theorem 3.15. Define $\lambda < 1$ to be the root of $1 - \lambda + 2\lambda \log \lambda = 0$ and define $\alpha = (1-\lambda)/2\sqrt{\log(1/\lambda)}$; we have $\lambda = 0.2846681370\ldots$ and $\alpha = 0.3190863431\ldots$. We will show that $c(F_t) = \big(\alpha + o(1)\big)t\sqrt{(\log t)/m(F)}$, where $m(F)$ is a function of the weighted graph F. Here the $o(1)$ term represents a quantity tending to zero as $t \to \infty$. This term is inevitable because the extremal graphs are related to random graphs, and all results are expressed in terms of large minors H or $t \to \infty$. The results are generally stated in the form 'given $\epsilon > 0$ there exists N such that for $n > N$ there is a minor with order at least $(1-\epsilon)$ times that required'; they could also be stated in the form 'given a sequence of graphs H_t with $|H_t| = t$, some function of these graphs has a particular limit'.

A general theme in this work is that, to find a minor H in a random graph, it suffices to find the sets of vertices W_u, for $u \in V(H)$, such that there is an edge between W_u and W_v whenever $uv \in E(H)$; given this, the sets W_u themselves can almost surely be made connected by using a few more vertices. In Section 2.2 we proved this for bipartite minors in random graphs by adapting the methods of Bollobás, Catlin and Erdős [2]. More generally, we can adapt the arguments of Thomason [64] that yield such a result for graphs that are sufficiently connected. In Section 3.2 we develop a general form of this method, which is then used in Section 3.4 and in Chapter 5. (This may also be used to derive results for random graphs; the random graphs are sufficiently connected by results of Bollobás and Thomason [3] that the connectivity of a random graph almost surely equals the minimum degree.)

The arguments of [64], presented in a more general form in Section 3.2, can then be used to make each part of the minor connected. This completes the results on minors of dense graphs. To complete the solution of the extremal function for F_t, arguments corresponding to those of Thomason [64] for large sparse G are also needed; those arguments give a K_{2t} minor, and he notes that the constant 2 could be replaced by any larger constant. Here we therefore develop a version of those arguments where the constant is arbitrary, so as to complete the proof for general blown-up minors.

Since the extremal graphs are derived from random graphs of a certain order and density, methods similar to those of Chapter 5 could be combined with those of this chapter to give a general description of all the extremal graphs as deriving from quasi-random graphs.

3.2 General arguments

Thomason [64] presents an argument showing that if a graph is reasonably connected, and partitions of most of the vertices can be found such that there are edges between the different parts of the partition, then a few vertices can be taken from the graph and used to make the parts of a partition of the rest of the graph connected, so yielding a minor. A variation of this argument is presented in [39]. Here we present a more general form of the method that can also be used when the minors to be found are not complete. We use the following lemma which is Lemma 4.1 of [64]; the simple proof may be found there.

Lemma 3.1 *Given a bipartite graph with vertex classes A and B, wherein each vertex of A has at least $\gamma|B|$ neighbours in B ($\gamma > 0$), there exists a set $M \subset B$ such that every vertex in A has a neighbour in M, and $|M| \leq \lfloor \log_{1/(1-\gamma)} |A| \rfloor + 1$.*

The following is the result it is the purpose of this section to prove.

Lemma 3.2 *Let $0 < \epsilon < 1$ be given. Then there exists N such that the following assertion holds.*

Let G be a graph of order $n > N$. Let H be a graph of order t with $n/\log n < t < n(\log\log n)/\sqrt{\log n}$. Let $C \subset V(G)$ be such that every pair u, $v \in V(G)$ have at least $n/(\log n)^{0.4}$ internally disjoint paths from u to v, of length at most $\log\log n$, the interiors of which lie entirely within C. Let $D \subset V(G) \backslash C$ be such that every vertex v of G has at least $n/10 \log\log n$ neighbours in D. Let $V(G) \setminus (C \cup D)$ be partitioned into sets W'_u, for $u \in V(H)$, such that there is an edge from W'_u to W'_v whenever $uv \in E(H)$. Then G has an H minor with parts W_u for $u \in V(H)$, where $W'_u \subset W_u$.

Proof By t applications of Lemma 3.1 we will find disjoint subsets $M_1, \dots,$ M_t in D such that every vertex of W_i' has a neighbour in M_i and $\sum_i |M_i| \leq 12n(\log\log n)^3/\sqrt{\log n}$, as follows. After $M_1, \dots, M_j$ have been chosen every vertex of $G - C - D$ has at least $n/10\log\log n - 12n(\log\log n)^3/\sqrt{\log n} > n/11\log\log n$ neighbours in D; so the conditions of that lemma apply with $A = W'_{j+1}$, $B = D \setminus \cup_{i=1}^{j} M_i$ and $\gamma = n/11(\log\log n)|D| \geq 1/(11\log\log n)$; so we have M_{j+1} with $|M_{j+1}| \leq 1 + \log_{1/(1-\gamma)}|A| \leq 1 + (\log|A|)/\gamma \leq 1 + 11(\log|A|)\log\log n$. Now we have $\sum_i |W_i'| < n$ and $\sum_i \log|W_i'|$ is maximised (for a given $\sum_i |W_i'|$) when the $|W_i'|$ are equal, so

$$\begin{aligned}\sum_i |M_i| &\leq t + 11(\log\log n)\sum_i \log|W_i'| \\ &\leq t + 11(\log\log n)t\log(n/t) \\ &\leq 12n(\log\log n)^3/\sqrt{\log n}\end{aligned}$$

for n sufficiently large.

We next find disjoint $N_1, \dots, N_t$ in C such that $M_i \cup N_i$ is connected (then, $W_i = W_i' \cup M_i \cup N_i$ will give an H minor). We can find such N_i with $|N_i| \leq |M_i|\log\log n$, since, given $N_1, \dots, N_j$, we have $|N_1 \cup \cdots \cup N_j| < 12n(\log\log n)^4/\sqrt{\log n}$ and we have $n/(\log n)^{0.4}$ paths of length at most $\log\log n$ with internal vertices in C between any pair of vertices u, v of M_{j+1}, so we find $|M_{j+1}| - 1$ such paths to connect M_{j+1}. □

The following standard results from [64], proved there, are used in conjunction with Lemma 3.2 to find the sets C and D. They are his Proposition 4.1 and Lemma 4.2 respectively.

Lemma 3.3 *Let $X \sim \mathrm{Bi}(n,p)$ be a binomially distributed random variable. Let $0 < \epsilon < 1$. Then $\mathbb{P}(|X - np| > \epsilon np) < 2e^{-\epsilon^2 np/4}$.*

Lemma 3.4 *Let G be a connected graph and let u, $v \in V(G)$. Then u and v are joined in G by at least $\kappa^2(G)/4|G|$ internally disjoint paths of length at most $2|G|/\kappa(G)$.*

3.3 Blown-up graphs as minors and critical graphs

This section represents joint work with Andrew Thomason.

In Section 2.2, we looked at complete bipartite minors and $K_s+\overline{K_t}$ minors. In this section, we generalise that work to a wider class of minors, which includes complete multipartite graphs as well, and find that the question of what graphs have an H minor for H in this class can be reduced to a particular subclass of *critical* minors. The class of H we consider is that of *blown-up* graphs, which we define as follows:

Definition 3.1 *A* weighted graph *is a pair $(F, \mathbf{w})$, where F is a graph with no vertices of degree zero,* where loops but not multiple edges are allowed, *and $\mathbf{w}$ is a vector of positive weights $w(v)$ for the vertices of F, with $\sum_{v \in V(F)} w(v) = 1$. We commonly refer to F (rather than the pair) as the weighted graph where this does not cause ambiguity. Then, for any positive integer t, a graph is a* blown-up graph F_t *if it is of order t and its vertices can be partitioned into disjoint sets $W(v)$ (for each $v \in V(F)$) such that $\lfloor w(v)t \rfloor \leq |W(v)| \leq \lceil w(v)t \rceil$ for all $v \in V(F)$ and such that there is an edge between $u_1 \in W(v)$ and $u_2 \in W(w)$ (with $u_1 \neq u_2$, but possibly $v = w$) if and only if there is an edge between v and w in F.*

Note that this definition does not quite specify F_t uniquely, because the exact size of each vertex set $W(v)$ is not precisely specified.

For a weighted graph F, write

$$m(F) = \max_{\substack{x(v):v\in V(F)\\ x(v)>0,\,\sum_{v\in V(F)} w(v)x(v)=1}} \min_{uv\in E(F)} x(u)x(v).$$

The quantity $m(F)$ turns out to determine the extremal function for when F_t is a minor of a random graph, as shown by the following theorem and Theorem 3.12 for minors in general graphs.

Theorem 3.5 *Let $0 < p < 1$ and write $q = 1 - p$. Let $0 < \epsilon < \frac{1}{2}$ be given. Then there exists N such that, if F is a weighted graph with all weights at least $n^{-\epsilon/3}$, and $n \geq N$, and $t = \left\lceil (1+\epsilon)\sqrt{m(F)}\, n/\sqrt{\log_{1/q} n}\right\rceil$, then the probability that any F_t is a minor of $\mathcal{G}(n,p)$ is at most ϵ.*

Proof Suppose $G = \mathcal{G}(n,p)$ has such a minor. Thus, the vertices of G may be partitioned into sets $T_{v,i}$ for $v \in V(F)$ and $1 \leq i \leq \lfloor w(v)t \rfloor$, such that there is an edge between distinct $T_{u,i}$ and $T_{v,j}$ whenever $uv \in E(F)$. Say a partition is *permissible* if it has such edges. There are at most n^n possible partitions. Let the probability that a given partition (of the fixed vertex set of order n, while G is random) is permissible be P. We then have

$$P = \prod_{(u,i)\neq(v,j),\, uv\in E(F)} \left(1 - q^{|T_{u,i}||T_{v,j}|}\right) \leq \exp\Bigg(-\sum_{(u,i)\neq(v,j),\, uv\in E(F)} q^{|T_{u,i}||T_{v,j}|}\Bigg).$$

The right hand side of this inequality is maximised when the sum is minimised. As before, holding all but $|T_{u,i}|$ and $|T_{u,j}|$ fixed, we see the minimum (if we allow the $|T_{u,i}|$ to take noninteger values) is where $|T_{u,i}| = |T_{u,j}|$. Suppose that at this minimum we have $|T_{u,i}| = x(u)n/t$. Observe that (for n sufficiently large) $\sum_{u\in V(F)} x(u)w(u) < 1+\epsilon^3$ and so $\min_{uv\in E(F)} x(u)x(v) < (1+\epsilon^3)m(F)$. Fix some u', v' achieving this minimum. We then have

$$P \;\leq\; \exp\Bigg[-\sum_{(u,i)\neq(v,j),\, uv\in E(F)} q^{x(u)x(v)n^2/t^2}\Bigg]$$

$$
\begin{aligned}
&< \exp\left[-\sum_{(u',i)\neq(v',j)} q^{(1+\epsilon^3)m(F)n^2/t^2}\right] \\
&\leq \exp\left[-\sum_{(u',i)\neq(v',j)} q^{(\log_{1/q} n)(1+\epsilon^3)/(1+\epsilon)^2}\right] \\
&\leq \exp\left[-\sum_{(u',i)\neq(v',j)} q^{(\log_{1/q} n)(1-\epsilon)}\right] \\
&= \exp\left[-\sum_{(u',i)\neq(v',j)} n^{-(1-\epsilon)}\right] \\
&\leq \exp\left[-(n^{-2\epsilon/3}/4)t^2 n^{-(1-\epsilon)}\right] \\
&\leq \exp\left[-\tfrac{1}{4}(1+\epsilon)^2 m(F) n^{1+\epsilon/3}/\log_{1/q} n\right].
\end{aligned}
$$

Since there are at most $n^n = \exp(n \log n)$ possible partitions, the probability that any one of them is permissible is less than ϵ for sufficiently large n.

□

The converse of this result follows from Theorem 3.12 on minors in general dense graphs, since the random graphs almost surely have sufficient connectivity.

We saw in Section 2.2 that a $K_{\beta t,(1-\beta)t}$ minor occurs in a random graph essentially just when a $K_{\beta t} + \overline{K_{(1-\beta)t}}$ minor does, if $\beta \leq \frac{1}{2}$. In the notation of this section, this means that a 2-vertex graph F with vertices u and v, and $w(u) = \beta \leq \frac{1}{2}$, with an edge uv and no edge vv, has the same $m(F)$ whether or not it has an edge uu. This extends naturally to a notion of *critical* graphs.

Definition 3.2 *A weighted graph F is said to be* critical *if adding any edge to F decreases $m(F)$, and merging any two vertices decreases $m(F)$. (When vertices are merged, the new vertex has as neighbourhood the union of the neighbourhoods of the old vertices, with itself as a neighbour if either old* *vertex had either old vertex as a neighbour, and the weight of the new vertex is the sum of the weights of the old vertices.)*

The following theorem characterises the critical graphs.

Theorem 3.6 *A weighted graph F of order t is critical if and only if its vertices can be ordered as v_1, v_2, …, v_t such that $v_iv_j \in E(F)$ if and only if $i+j>t$, and*

$$\frac{w(v_t)}{w(v_1)} < \frac{w(v_{t-1})}{w(v_2)} < \cdots < \frac{w(v_{t+1-\lfloor t/2 \rfloor})}{w(v_{\lfloor t/2 \rfloor})} < 1.$$

This graph has

$$m(F) = \left[\sum_{i=1}^{t} \sqrt{w(v_i)w(v_{t+1-i})}\right]^{-2}.$$

Proof Suppose F is critical. Suppose that $x(v)$ are assigned such that we have $m(F) = \min_{uv \in E(F)} x(u)x(v)$. Say that an edge uv of F is *critical* if $x(u)x(v) = m(F)$.

Every vertex must be adjacent to a critical edge; if u were not, $x(u)$ could be slightly decreased and $x(v)$ slightly increased for all vertices $v \neq u$. No distinct vertices u and v can have $x(u) = x(v)$; for such vertices could be merged without affecting $m(F)$. This means that every vertex is adjacent to exactly one critical edge (for, if uv and uw were critical, we would have $x(v) = x(w)$), and that there is at most one critical loop (for, if uu and vv were critical, we would have $x(u) = x(v)$).

Each critical edge uv must have one endpoint u with $x(u) < \sqrt{m(F)}$, and the other v with $x(v) > \sqrt{m(F)}$, except that a critical loop uu must have $x(u) = \sqrt{m(F)}$. Let v_1, v_2, …, $v_{\lfloor t/2 \rfloor}$ be the vertices with $x(v_i) < \sqrt{m(F)}$, in increasing order of $x(v_i)$. Let $v_{(t+1)/2}$ be the vertex of the critical loop, if t is odd. Let v_i and v_{t+1-i} be the endpoints of a critical edge, for $1 \leq i \leq$

$\lfloor t/2 \rfloor$. Since $x(v_{t+1-i})x(v_i) = m$, it follows that $x(v_1) < x(v_2) < \cdots < x(v_t)$. Criticality of F means that each v_i has as neighbours exactly those v_j for which $x(v_i)x(v_j) \geq m$; that is, $i + j > t$. Thus F has exactly the edges described.

Suppose now that $v_i v_j$ is a critical edge. We have $m(F) = x(v_i)x(v_j) = w(v_i)x(v_i)w(v_j)x(v_j)/w(v_i)w(v_j)$. If we hold $x(v)$ fixed for all $v \neq v_i$, v_j, then $x(v_i)$ and $x(v_j)$ may be varied such that $w(v_i)x(v_i)+w(v_j)x(v_j)$ remains constant. Thus $x(v_i)x(v_j)$ is locally maximised when $w(v_i)x(v_i) = w(v_j)x(v_j)$, which must hold by criticality. This means that $x(v_j) = w(v_i)x(v_i)/w(v_j)$, so that

$$m(F) = x(v_i)x(v_j) = x(v_i)^2 w(v_i)/w(v_j);$$

so

$$x(v_i)^2 = m(F)w(v_j)/w(v_i) = m(F)w(v_{t+1-i})/w(v_i).$$

The condition the theorem gives on the weights now follows from $x(v_1) < x(v_2) < \cdots < x(v_{\lfloor t/2 \rfloor}) < \sqrt{m(F)}$.

Since $m(F) = x(v_i)^2 w(v_i)/w(v_{t+1-i})$, we have

$$w(v_i)x(v_i) = \sqrt{m(F)w(v_i)w(v_{t+1-i})}.$$

Since

$$\sum_{i=1}^{t} w(v_i)x(v_i) = 1,$$

we conclude that $m(F)$ has the value given in the theorem.

Conversely, suppose that F is a weighted graph of the described form. We need to show that F is critical. Suppose again that $x(v)$ are assigned such that we have $m(F) = \min_{uv \in E(F)} x(u)x(v)$. Supposing that F is not critical, this assignment may be chosen so that some edge can be added, or some two vertices merged, keeping the assignment and without decreasing $m(F)$, as follows. We take an assignment for the graph with added edge or merged vertices, and pull back that assignment to one for F. If an edge was added, it need just be removed. If two vertices were merged, split the resulting vertex into two vertices with identical neighbourhoods, with each of them receiving the $x(v)$ of the merged vertex, then remove edges so that the edge set of the graph is again that of F. Again, every vertex must be adjacent to a critical edge.

For $i < j$, we have $\Gamma(v_i) \subset \Gamma(v_j)$. Thus $x(v_i) \leq x(v_j)$; so, since v_i is adjacent to at least one critical edge, the edge $v_i v_{t+1-i}$ must be critical for all i. If some other edge $v_i v_{t+1-i+k}$ is critical, we then have $x(v_{t+1-i}) = x(v_{t+1-i+k})$ and so $x(v_i) = x(v_{i-k})$. We shall now show that the $x(v_i)$ are strictly increasing, with $x(v_i)w(v_i) = x(v_{t+1-i})w(v_{t+1-i})$ for all i, by showing that otherwise the $x(v_j)$ could be varied so as to increase $m(F)$.

Let $i \leq t/2$, let $x(v_i)$ be first in its sequence of consecutive equal $x(v_j)$, and let $x(v_{i'})$ be last, so v_i and $v_{t+1-i'}$ are each adjacent to only one critical edge. If $x(v_i)w(v_i) \neq x(v_{t+1-i})w(v_{t+1-i})$, consider varying $x(v_i)$ and $x(v_{t+1-i})$ to make them more nearly equal, then reducing them while increasing the other $x(v_j)$. Unless $i < i'$ and this involves increasing $x(v_i)$ and decreasing $x(v_{t+1-i})$, we conclude that $x(v_i)w(v_i) = x(v_{t+1-i})w(v_{t+1-i})$; in the remaining case $x(v_i)w(v_i) < x(v_{t+1-i})w(v_{t+1-i})$, so in any case if $i < i'$ we have $x(v_i)w(v_i) \leq x(v_{t+1-i})w(v_{t+1-i})$; thus $i' < t/2$ and likewise $x(v_{i'})w(v_{i'}) \geq x(v_{t+1-i'})w(v_{t+1-i'})$; but this implies $w(v_{t+1-i})/w(v_i) \geq x(v_i)/x(v_{t+1-i}) = x(v_{i'})/x(v_{t+1-i'}) \geq w(v_{t+1-i'})/w(v_{i'})$, a contradiction for $i < i'$. Thus the $x(v_i)$ are strictly increasing with $x(v_i)w(v_i) = x(v_{t+1-i})w(v_{t+1-i})$ for all i.

This, however, means that adding any edge will decrease $m(F)$. Since the $x(v_i)$ are distinct, no two vertices may be merged either (through how we derived the assignment for a noncritical graph in which vertices could be merged by pulling back one for the graph with those vertices merged). Thus F is critical. □

As a simple example, consider the case where F_t is complete multipartite; that is, F is a complete graph (with no loops). Some process of adding edges and merging vertices yields a critical weighted graph from F. Any critical graph on more than 2 vertices has two distinct vertices with no edge between them, which cannot arise in this way from a complete F. Thus the critical graph corresponding to F has 1 or 2 vertices; if it has 2 vertices, the vertex with no edge to itself has more than half the weight and must have arisen from such a vertex in the original F, with which no other vertex has been merged. Thus we see that the critical graph corresponding to complete F (complete multipartite F_t) has 1 or 2 vertices, and has 2 vertices if and only if F has a vertex with weight more than $\frac{1}{2}$.

We have seen that how large an F_t minor is present in a random graph depends (up to a $1+o(1)$ factor) only on $m(F)$. We shall see in Theorem 3.12 that the same applies for minors in general graphs. This means that, for any weighted graph F, there is a critical graph F' such that F_t is a minor of a random graph essentially (that is, up to a $1 + o(1)$ factor in t) just when F'_t is. We now consider more general minors than blown-up graphs. We shall see that any graph is a subgraph (plus a few edges) of a blow-up of a much smaller critical graph that is no harder to find as a minor of $\mathcal{G}(n, p)$. However, in some cases the question of whether H is a minor of $\mathcal{G}(n, p)$ turns out to depend on the few edges not in this blown-up graph, and we do not have a fully general theory of sparse graphs as minors although we show in [41] that the extremal graphs for dense H are random and that regular graphs H of order t and size $t^{2-\beta}$ cannot be found as minors if $n < t\sqrt{(1-\beta)\log_{1/q} t}$.

Theorem 3.7 *Let $0 < \epsilon < \frac{1}{4}$ be given. Then there exists N such that the* *following assertion holds.*

Let $\epsilon < p < 1-\epsilon$. Let $n > N$. Let H be a graph of edge density greater than ϵ, and write $t = |H|$. Suppose that

$$\mathbb{P}\big(H \prec \mathcal{G}(n,p)\big) > \epsilon.$$

Then there exists a critical weighted graph F with at most $(7/\epsilon)\log t$ vertices and with all weights at least $t^{-\epsilon/6}$ such that there is some F_t on the same vertex set as H with H having at most $t^{2-\epsilon/10}$ edges not in F_t, and $\sqrt{m(F)}\, n/\sqrt{\log_{1/q} n} > (1-\epsilon)t$.

Proof If $t \le (1+\epsilon)n/\sqrt{\log_{1/q} n}$ then taking F to be the critical graph on a single vertex will suffice, so suppose $t > (1+\epsilon)n/\sqrt{\log_{1/q} n}$. Since $\mathbb{P}\big(H \prec \mathcal{G}(n,p)\big) > \epsilon$, there is some choice of $y(u)$ for $u \in V(H)$ such that, for any given partition of the vertex set of $G = \mathcal{G}(n,p)$ into t parts W_u of orders $y(u)n/t$, the probability that that partition has an edge between W_u and W_v whenever $uv \in E(H)$ is at least ϵn^{-n}. Fix such $y(u)$. Note that the sum of the $y(u)$ is t. We have

$$\epsilon n^{-n} \le \prod_{uv \in E(H)} (1 - q^{|W_u||W_v|}) \le \exp\Bigg(-\sum_{uv\in E(H)} q^{|W_u||W_v|}\Bigg)$$

so that

$$\sum_{uv\in E(H)} q^{|W_u||W_v|} = \sum_{uv\in E(H)} q^{y(u)y(v)n^2/t^2} \le n\log n - \log\epsilon.$$

We divide the vertices of H up according to the value of $y(u)$. For each integer k put

$$A_k = \left\{\, u \in V(H) : (1-\epsilon/3)^{k+1/2} \le y(u) \le (1-\epsilon/3)^{k-1/2} \,\right\}.$$

For all u we have $t/n \le y(u) \le t$. Thus there are at most $2\log_{1/(1-\epsilon/3)} t < (7/\epsilon)\log t$ nonempty A_k. Writing $e(A_i, A_j)$ for the number of edges between

A_i and A_j, or for the number within A_i when $i = j$, we have

$$\sum_{i,j} e(A_i, A_j) q^{(1-\epsilon/3)^{i+j-1} n^2/t^2} \leq n \log n - \log \epsilon$$

where the sum is over unordered pairs of i and j. If $t/n = \sqrt{m}/\sqrt{\log_{1/q} n}$, where $\sqrt{m} > (1+\epsilon)$, this means that

$$\sum_{i,j} e(A_i, A_j) n^{-(1-\epsilon/3)^{i+j-1}/m} \leq n \log n - \log \epsilon.$$

Say $m = (1-\epsilon/3)^{-r}$, where $r \geq 1$. If $i+j+r-1 \geq 1$, then $n^{-(1-\epsilon/3)^{i+j-1}/m} \geq n^{-1+\epsilon/3}$. If $e(A_i, A_j) \geq t^{2-\epsilon/4}$, we then have a contradiction. Thus $e(A_i, A_j) < t^{2-\epsilon/4}$ for all i, j with $i+j+r-1 \geq 1$.

We now derive a blown-up graph from our given H. First discard all edges from A_i to A_j where $i+j+r-1 \geq 1$; this is at most $t^{2-\epsilon/5}$ edges. Then discard all vertices in the A_i with $|A_i| \leq t^{1-\epsilon/6}$; this loses at most $t^{2-\epsilon/7}$ edges. Also discard any A_i with no edges from them; because of the density requirement, not all A_i are discarded. Now add all edges from A_i to A_j where $i+j+r-1 < 1$. This yields a blown-up graph, and so a corresponding weighted graph H^* with at most $(7/\epsilon)\log t$ vertices and with all weights at least $t^{-\epsilon/6}$, and we see that H has at most $t^{2-\epsilon/10}$ edges not in H^*_t. (Discarding the A_i with no edges from them was necessary to ensure the absence of vertices of degree zero in H^*.) Also, by construction of H^*, using the x-values $(1-\epsilon/3)^{k+1/2}$ for A_k (possibly slightly scaled up to give $\sum x(u)w(u) = 1$), we see that $\sqrt{m(H^*)}\, n/\sqrt{\log_{1/q} n} > (1-\epsilon)t$. Now take the critical graph F containing H^*, using Theorem 3.6. □

One specific case of this is worthy of note. This is the question of whether a graph is any easier to find as a minor than a complete graph of the same order. The property that determines this in one direction is that of whether a graph has a *tail*; roughly, whether it is a subgraph (plus a few edges) of a blow-up of a critical graph of order 2. For a graph without that property, the relevant critical graph is that with a single vertex. (The above comments about sparse graphs that are hard to find as minors mean the converse cannot be so simple.) These results are formalised in what follows.

Definition 3.3 *Let H be a graph of order t. An ϵ-tail in H is an ordered pair (S,T) of disjoint subsets of $V(H)$ with $|S| < |T| - \epsilon t$ and $e(T, V(H)\backslash S) \leq t^{2-\epsilon}$.*

Note that a ϵ-tail is also an η-tail for any $\eta \leq \epsilon$.

Theorem 3.8 *Let $0 < \epsilon < \frac{1}{9}$ and $0 < p < 1$ be given and put $q = 1 - p$. Then there exists t_0 such that if $t > t_0$, $n = \left\lceil (1 - \sqrt{\epsilon}\,)\, t \sqrt{\log_{1/q} t}\, \right\rceil$ and H is a graph of order t with no ϵ-tail, then $H \prec \mathcal{G}(n,p)$ with probability at most ϵ.*

Proof If $H \prec G$, then as usual we have a partition of $V(G)$ into W_u for $u \in V(H)$, with an edge between W_u and W_v whenever $uv \in E(H)$, and there must be some choice of the $|W_u|$ such that the probability that a given partition with those W_u has such edges is at least ϵn^{-n}. Thus we have

$$\epsilon n^{-n} \leq \prod_{uv \in E(H)} (1 - q^{-|W_u||W_v|}) \leq \exp\biggl(- \sum_{uv \in E(H)} q^{|W_u||W_v|} \biggr).$$

We will show that $\sum_{uv \in E(H)} q^{|W_u||W_v|} \geq t^{1+\epsilon/2}$, which yields a contradiction.

Write $n = t\ell$ where $\ell = (1 - \sqrt{\epsilon}\,) \sqrt{\log_{1/q} t}$. As in the proof of the previous theorem we divide the vertices of H into classes A_k, but here it is more convenient to use linear rather than exponential bounds for those classes: we set

$$A_k = \left\{ u \in V(H) : \left(1 - (k + \tfrac{1}{2})\sqrt{\epsilon}\,\right) \ell \leq |W_u| \leq \left(1 - (k - \tfrac{1}{2})\sqrt{\epsilon}\,\right) \ell \right\}.$$

We then have $\sum_k |A_k| = t$ and A_k empty when $(k-\frac{1}{2})\sqrt{\epsilon} \geq 1$. Again we consider which A_i have edges to which A_j. For $u \in V(H)$ define $\sigma(u)$ by $u \in A_{\sigma(u)}$. If for some u, $v \in V(H)$ we have $\sigma(u) = i$ and $\sigma(v) = j$ with $i+j \geq -1$, we then have $|W_u||W_v|/\ell^2 \leq \left(1-(i-\frac{1}{2})\sqrt{\epsilon}\,\right)\left(1-(j-\frac{1}{2})\sqrt{\epsilon}\,\right) = 1-(i+j-1)\sqrt{\epsilon}+(i-\frac{1}{2})(j-\frac{1}{2})\epsilon$. For a given i with $(i-\frac{1}{2})\sqrt{\epsilon} < 1$, we see that $|W_u||W_v|/\ell^2$ is maximised when j is minimised; that is, when $i+j=-1$, and subject to $i+j=-1$ the expression is then maximised when i and j are 0 and -1 in some order. Thus for any u and v with $\sigma(u)+\sigma(v) \geq -1$, we have $|W_u||W_v| \leq (1+\frac{3}{2}\sqrt{\epsilon}\,)(1+\frac{1}{2}\sqrt{\epsilon}\,)\ell^2 \leq (1+\sqrt{\epsilon}\,)^2\ell^2$.

If there are at least $t^{2-\epsilon}$ edges $uv \in E(H)$ with $\sigma(u)+\sigma(v) \geq -1$, we then have $\sum_{uv\in E(H)} q^{|W_u||W_v|} \geq t^{2-\epsilon}q^{(1+\sqrt{\epsilon})^2\ell^2} = t^{2-\epsilon-(1+\sqrt{\epsilon})^2(1-\sqrt{\epsilon})^2} = t^{2-\epsilon-(1-\epsilon)^2} \geq t^{1+\epsilon/2}$, as required.

We now suppose that there are fewer than $t^{2-\epsilon}$ edges $uv \in E(H)$ with $\sigma(u)+\sigma(v) \geq -1$, and show for a contradiction that H has an ϵ-tail. We consider some unions of the A_i: for $m \geq 0$ put $T_m = \cup_{k\geq m} A_k$ and $S_m = \cup_{k<-m-1} A_k$. If $u \in T_m$ and $v \notin S_m$ we have $\sigma(u)+\sigma(v) \geq -1$, so that $e\big(T_m, V(H)-S_m\big) < t^{2-\epsilon}$. For there not to be an ϵ-tail we must then have $|S_m| \geq |T_m| - \epsilon t$.

Since $\sum_u |W_u| = n = t\ell$, we have

$$\sum_k |A_k|\left(1-(k+\tfrac{1}{2})\sqrt{\epsilon}\,\right)\ell \leq t\ell \leq \sum_k |A_k|\left(1-(k-\tfrac{1}{2})\sqrt{\epsilon}\,\right)\ell.$$

Since $\sum_k |A_k| = t$ we conclude that

$$-\frac{t}{2} \leq \sum_k k|A_k| \leq \frac{t}{2}.$$

But we also have

$$\sum_k k|A_k| \;=\; \sum_{i\geq 1} |T_i| - |A_{-1}| - 2|S_0| - \sum_{i\geq 1} |S_i|$$

$$
\begin{aligned}
&= \sum_{1\le i<1/2+1/\sqrt{\epsilon}} |T_i| - |A_{-1}| - 2|S_0| - \sum_{i\ge 1} |S_i| \\
&\le (1/\sqrt{\epsilon}+\tfrac{1}{2})\epsilon t - |A_{-1}| - 2|S_0|.
\end{aligned}
$$

Now $t = \sum_k |A_k| = |S_0|+|A_{-1}|+|T_0| \le 2|S_0|+|A_{-1}|+\epsilon t$, so $-|A_{-1}|-2|S_0| \le (-1+\epsilon)t$. Thus $\sum_k k|A_k| \le (-1+\epsilon+\sqrt{\epsilon}+\epsilon/2)t < -t/2$, a contradiction. □

3.4 Minors of dense graphs

In this section, we suppose that we have some fixed weighted graph F. Suppose also that we have some positive $x(v)$ for $v \in V(F)$, with

$$\sum_{v\in V(F)} w(v)x(v) = 1,$$

and write

$$m(\mathbf{x}) = \min_{uv\in E(F)} x(u)x(v).$$

(It will be best to choose $\mathbf{x}$ so that $m(\mathbf{x}) = m(F)$, but the arguments in this section do not all require that choice.)

The main purpose of this section is to prove Theorem 3.12, that a graph of order n, density $1-q$ and reasonable connectivity has an F_t minor, for $t = (1-\epsilon)\sqrt{m(\mathbf{x})}\, n/\sqrt{\log_{1/q} n}$; in view of Theorem 3.5, this is best possible. To do this, we will show that a partition into t parts can be found, with the necessary edges between the parts, and then Lemma 3.2 will be applied to find the minor.

To find the partition, we generalise the method of equipartitions used by Thomason [64]. Let some F_t be fixed, and suppose that in this F_t there are $w^*(v)$ vertices corresponding to $v \in V(F)$, where $\lfloor w(v)t\rfloor \le w^*(v)t \le \lceil w(v)t\rceil$ for all $v \in V(F)$. Define an $(F_t,\mathbf{x})$-*equipartition* of G to be a partition of $V(G)$ into t sets W_v for $v \in V(F_t)$, where there is an edge between W_u and W_v whenever $uv \in E(F_t)$, where the parts corresponding to a vertex $v \in V(F)$ have total order $\lfloor x(v)w(v)|G| \rfloor$ or $\lceil x(v)w(v)|G| \rceil$, a quantity we denote $x^*(v)w^*(v)|G|$, and where each of the $w^*(v)t$ parts corresponding to $v \in V(F)$ has order $\lfloor x^*(v)|G|/t \rfloor$ or $\lceil x^*(v)|G|/t \rceil$.

The method of equipartitions used in [64] finds the equipartition as a suitably constrained random partition; if the partition were chosen at random, there could be too many parts with small neighbourhoods, but if it is chosen in a suitably constrained way, so that each part has an even distribution of vertex degrees, this does not occur. The key adaptation of this method needed for it to be used to find blown-up minors is to divide the vertices of G *at random* into the parts for each $u \in V(F)$, before applying the method to find our constrained random partition. For this to work, we need to use a result of Chvátal [10] on the tail of the hypergeometric distribution:

Lemma 3.9 (Chvátal [10]) *Let*

$$H(M, N, n, k) = \sum_{i=k}^{n} \binom{M}{i} \binom{N-M}{n-i} \binom{N}{n}^{-1}.$$

Let $p = M/N$ and suppose $k = (p+t)n$ for some $0 \le t \le 1-p$. Then

$$H(M, N, n, k) \le \left(\left(\frac{p}{p+t} \right)^{p+t} \left(\frac{1-p}{1-p-t} \right)^{1-p-t} \right)^{n} \le e^{-2t^2 n}.$$

The result achieved on equipartitions is as follows:

Lemma 3.10 *Let G be a graph of order $n > 10000|F|$ and edge density $p = 1-q$. Let ℓ and s be positive integers with $\ell \ge 2$ and $s = \lfloor n/\ell \rfloor \ge 2$. Let some graph F_s be fixed. Let positive $w^*(v)$ be as defined above, let some positive $x(v)$ for $v \in V(F)$ be given with $\sum_v w(v)x(v) = 1$, and let some $x^*(v)$ as above be chosen (that is, $x^*(v)w^*(v)|G|$ an integer for all $v \in V(F)$,*

$\sum_v x^*(v)w^*(v) = 1$ *and* $\lfloor x(v)w(v)|G|\rfloor \leq x^*(v)w^*(v)|G| \leq \lceil x(v)w(v)|G|\rceil$*). Let* $\beta = n^{-1/4}$*. Let* $\beta' = \beta n/\min_{u\in V(F)}\big(\lfloor x^*(u)\ell\rfloor w^*(u)s\big)$*. Let* $0 < \eta \leq p - 2\beta'$ *and* $\omega \geq 1$*. Suppose that* $\ell x^*(u) \geq 2$ *for all* $u \in V(F)$*. Write*

$$E_1 = \frac{3|F|s}{\omega\eta}$$

and

$$E_2 = 3s^2 \sum_{uv\in E(F)} (6\omega)^{\ell\max\{x^*(u),x^*(v)\}} \left[\frac{q+2\beta'}{1-\eta}\right]^{(1-\eta)\lfloor \ell x^*(u)\rfloor(\lfloor \ell x^*(v)\rfloor - 1)}.$$

Then G *has an* $(F_t, \mathbf{x})$*-equipartition for some* t *with*

$$t \geq s - \frac{E_1 + E_2 + 1}{\min_{u\in V(F)} w(u)}.$$

Proof First take a random partition of $V(G)$ into sets V_u for $u \in V(F)$, where $|V_u| = x^*(u)w^*(u)n$. We claim that this partition can be taken so that, for all $u \in V(F)$ and $v \in V(G)$,

$$\big||\Gamma(v) \cap V_u| - x(u)w(u)d(v)\big| < n^{3/4}.$$

We have $|x(u)w(u)d(v) - x^*(u)w^*(u)d(v)| \leq 1$ and $n^{3/4} - 1 > n^{3/4}/2$. Now,

$$\mathbb{P}\left(\big||\Gamma(v) \cap V_u| - x^*(u)w^*(u)d(v)\big| \geq tx^*(u)w^*(u)n\right) \leq 2e^{-2t^2x^*(u)w^*(u)n}$$

by Lemma 3.9, for any $t \geq 0$. Take $t = n^{-1/4}/2x^*(u)w^*(u)$. We have $2t^2x^*(u)w^*(u)n \geq n^{1/2}/2$ so that $2e^{-2t^2x^*(u)w^*(u)n} < n^{-4} < n^{-3}/3|F|$. Thus the probability that

$$\big||\Gamma(v) \cap V_u| - x(u)w(u)d(v)\big| \geq n^{3/4}$$

is less than $n^{-3}/3|F|$, and so the probability that this holds for any u and v is at most $1/3n^2$. Thus some partition in which each vertex has (to within βn) the expected number of neighbours in each V_u may be taken. We may also take a random $V'_u \subset V_u$ with $|V'_u| = w^*(u)s\lfloor x^*(u)\ell\rfloor$ and arrange for all vertices to have within βn of the expected number of neighbours within each V'_u. Fix such a partition in what follows.

For $u \in V(F)$ and $v \in V(G)$, let

$$Q(V_u, v) = \big\{\, w \in V_u - \{v\} : vw \notin E(G) \,\big\}$$

be the set of nonneighbours (other than v) of v within V_u, let $Q(V'_u, v) = Q(V_u, v) \cap V'_u$, let

$$Q(v) = \bigcup_{u \in V(F)} Q(V_u, v)$$

and for $W \subset V(G)$ let

$$N(V'_u, W) = \{\, v \in V'_u : W \subset Q(v) \,\}$$

be the set of vertices in $V'_u - W$ with no edge to W.

For each $u \in V(F)$, order the vertices of V'_u as $v(V'_u, 1)$, $v(V'_u, 2)$, $\dots$, $v(V'_u, |V'_u|)$ in increasing order of $\big|Q\big(v(V'_u, i)\big)\big|$; that is, in decreasing order of degree in G. Write $q(V'_u, i) = \big|Q\big(v(V'_u, i)\big)\big|/(n-1)$. Now divide V'_u into $\lfloor x^*(u)\ell\rfloor$ *blocks*

$$B(V_u, j) = \{\, v(V'_u, i) : (j-1)w^*(u)s < i \le jw^*(u)s \,\}$$

each of order $w^*(u)s$, for $1 \le j \le \lfloor x^*(u)\ell\rfloor$. For $1 \le j \le \lfloor x^*(u)\ell\rfloor$ now choose a random permutation $\beta(V_u, j)$ of B_j, these permutations being chosen independently and uniformly at random from all $\big(w^*(u)s\big)!$ permutations, and so derive a constrained random partition of $\{\, v(V'_u, i) : 1 \le i \le w^*(u)s\lfloor x^*(u)\ell\rfloor \,\}$ into $w^*(u)s$ parts

$$W(V_u, i) = \big\{\, v\big(V'_u, \beta(V_u, j)(i)\big) : 1 \le j \le \lfloor x^*(u)\ell\rfloor \,\big\}$$

each of $\lfloor x^*(u)\ell\rfloor$ vertices.

For $S \subset V'_u$ and W one of the random parts $W(V_u, i)$, we have

$$\begin{aligned}
\mathbb{P}(W \subset S) &= \prod_{j=1}^{\lfloor x^*(u)\ell \rfloor} \frac{|S \cap B_j|}{w^*(u)s} \\
&\le \left[\frac{1}{\lfloor x^*(u)\ell \rfloor} \sum_{j=1}^{\lfloor x^*(u)\ell \rfloor} \frac{|S \cap B_j|}{w^*(u)s} \right]^{\lfloor x^*(u)\ell \rfloor} \\
&= \left[\frac{|S|}{\lfloor x^*(u)\ell \rfloor w^*(u)s} \right]^{\lfloor x^*(u)\ell \rfloor} .
\end{aligned}$$

Taking $S = Q\big(V'_u, v(V'_v, i)\big)$, we have

$$\mathbb{P}\big(v(V'_v, i) \in N(V'_v, W)\big) = \mathbb{P}(W \subset S) \le \big(q(V'_v, i) + \beta'\big)^{\lfloor x^*(u)\ell \rfloor}.$$

We thus have

$$\mathbb{E}\big(|B(V_v, j) \cap N(V'_v, W)|\big) \le w^*(v)s\left(q\big(V'_v, jw^*(v)s\big) + \beta'\right)^{\lfloor x^*(u)\ell \rfloor}.$$

Say that a random part W in V_u *rejects* a block $B(V_v, j)$ if $|B(V_v, j) \cap N(V'_v, W)| > \omega w^*(v)s\left(q\big(V'_v, jw^*(v)s\big) + \beta'\right)^{\lfloor x^*(u)\ell \rfloor}$, so that W rejects any given block $B(V_v, j)$ with probability at most $1/\omega$. Now write

$$R(V_v, W) = \{\, j < \lfloor x^*(v)\ell \rfloor : W \text{ rejects } B(V_v, j)\,\}.$$

We have $\mathbb{E}\big(|R(V_v, W)|\big) \le \big(\lfloor x^*(v)\ell \rfloor - 1\big)/\omega$; say that a random part W is *acceptable* if $|R(V_v, W)| < \eta\big(\lfloor x^*(v)\ell \rfloor - 1\big)$ for all $v \in V(F)$, so the probability that a random part W is not acceptable is at most $|F|/\omega\eta$.

Now let W be some given acceptable part in V'_u, and let $v \in V(F)$ be such that $uv \in E(F)$. Let $M(V_v, W) = \{1, 2, \ldots, \lfloor x^*(v)\ell \rfloor - 1\} \setminus R(V_v, W)$ and let $m(V_v) = |M(V_v, W)| \ge (1 - \eta)\big(\lfloor x^*(v)\ell \rfloor - 1\big)$. Let W' be some random part in V'_v that is not equal to W. Let P_W be the probability conditional on W that there is no edge between W and W'. Let $I = 1$ if $u = v$ and let $I = 0$ otherwise. We then have

$$P_W = \mathbb{P}\big(W' \subset N(V'_v, W) \bigm| W\big)$$

$$\begin{aligned}
&\leq \prod_{j\in M(V_v,W)} \frac{\omega w^*(v)s\left(q\big(V_v',jw^*(v)s\big)+\beta'\right)^{\lfloor x^*(u)\ell\rfloor}}{w^*(v)s-I}\\
&\leq (2\omega)^{x^*(v)\ell}\prod_{j\in M(V_v,W)}\left(q\big(V_v',jw^*(v)s\big)+\beta'\right)^{\lfloor x^*(u)\ell\rfloor}.
\end{aligned}$$

Now observe that

$$\begin{aligned}
\sum_{j\in M(V_v,W)} w^*(v)s\left(q\big(V_v',jw^*(v)s\big)+\beta'\right) &\leq \sum_{i=1}^{|V_v'|}\big(q(V_v',i)+\beta'\big)\\
&\leq (q+2\beta')\big(w^*(v)s\lfloor x^*(v)\ell\rfloor\big).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\left[\prod_{j\in M(V_v,W)}\Big(q\big(V_v',jw^*(v)s\big)+\beta'\Big)\right]^{1/m(V_v)}\\
&\qquad\leq \frac{1}{m(V_v)}\sum_{j\in M(V_v,W)}\Big(q\big(V_v',jw^*(v)s\big)+\beta'\Big)\\
&\qquad\leq \frac{(q+2\beta')\big(w^*(v)s\lfloor x^*(v)\ell\rfloor\big)}{m(V_v)w^*(v)s}\\
&\qquad= \frac{\lfloor x^*(v)\ell\rfloor}{m(V_v)}(q+2\beta')\\
&\qquad\leq \frac{\lfloor x^*(v)\ell\rfloor}{\lfloor x^*(v)\ell\rfloor-1}\times\frac{q+2\beta'}{1-\eta}.
\end{aligned}$$

Since $q+2\beta'\leq 1-\eta$ and $m(V_v)\geq(1-\eta)\big(\lfloor x^*(v)\ell\rfloor-1\big)$, it follows that

$$\begin{aligned}
P_W &\leq (2\omega)^{x^*(v)\ell}\left[\frac{\lfloor x^*(v)\ell\rfloor}{\lfloor x^*(v)\ell\rfloor-1}\times\frac{q+2\beta'}{1-\eta}\right]^{m(V_v)\lfloor x^*(u)\ell\rfloor}\\
&\leq (2\omega)^{x^*(v)\ell}\left[\frac{\lfloor x^*(v)\ell\rfloor}{\lfloor x^*(v)\ell\rfloor-1}\right]^{\lfloor x^*(u)\ell\rfloor(\lfloor x^*(v)\ell\rfloor-1)}\\
&\quad\times\left[\frac{q+2\beta'}{1-\eta}\right]^{(1-\eta)\lfloor x^*(u)\ell\rfloor(\lfloor x^*(v)\ell\rfloor-1)}\\
&\leq (6\omega)^{\ell\max\{x^*(u),x^*(v)\}}\left[\frac{q+2\beta'}{1-\eta}\right]^{(1-\eta)\lfloor \ell x^*(u)\rfloor(\lfloor \ell x^*(v)\rfloor-1)}.
\end{aligned}$$

It now follows that the probability that a partition has more than E_1 unacceptable parts is less than $\frac{1}{2}$, as is the probability that a partition has more than E_2 pairs of acceptable parts without edges between them. So take a partition with no more than E_1 unacceptable parts and no more than E_2 defective pairs, and remove the unacceptable parts and one part from each defective pair. This might have removed disproportionately many parts corresponding to some vertex $u \in V(F)$, but the number of parts corresponding to each such u can be made balanced again with the removal in total of not more than $(E_1 + E_2 + 1)/\big(\min_{u \in V(F)} w(u)\big)$ parts. □

The above formulation is somewhat cumbersome to work with. To obtain a more useful form, we will take $s \approx \sqrt{m(\mathbf{x})}\, n/\sqrt{\log_{1/q} n}$ and $\ell \approx \sqrt{\log_{1/q} n}/\sqrt{m(\mathbf{x})}$ and show that E_1 and E_2 are small compared to s. The result we obtain is the following.

Corollary 3.11 *Let $0 < \epsilon < 1$. Let a weighted graph F and some $\mathbf{x}$ as above be given. Then there exists N such that the following assertion holds.*

Let G be a graph of order $n > N$ and edge density $p = 1 - q$, where $(\log \log n)^{2+\epsilon}/\log n < p < 1 - (\log n)^{-1/\epsilon}$. Then G has an $(F_t, \mathbf{x})$-equipartition for some t with $t \geq (1 - \epsilon)\sqrt{m(\mathbf{x})}\, n/\sqrt{\log_{1/q} n}$.

Proof Suppose throughout that n is sufficiently large for various statements in this proof to work. Write $w = \min_{u \in V(F)} w(u)$ and $x = \max_{u \in V(F)} x(u)$. We have $q > (\log n)^{-1/\epsilon}$, so

$$\log_{1/q} n = (\log n)/\big(\log(1/q)\big) \to \infty$$

as $n \to \infty$. Put

$$\ell = \left\lceil (1 + \epsilon/2)\sqrt{\log_{1/q} n}\Big/\sqrt{m(\mathbf{x})} \right\rceil,$$

$\eta = \epsilon p/8$ and $\omega = 256|F|/\epsilon^2 wp$. Then we have

$$s = \lfloor n/\ell \rfloor > (1 - \epsilon/2)\sqrt{m(\mathbf{x})}\, n\Big/\sqrt{\log_{1/q} n}.$$

We will show that (where E_1 and E_2 are as in Lemma 3.10)

$$\frac{E_1 + E_2 + 1}{w} \leq \epsilon s/2$$

which in conjunction with Lemma 3.10 implies our result. Clearly $1/w < \epsilon s/6$, and by choice of ω and η we have $E_1/w = 3|F|s/w\omega\eta < \epsilon s/6$. So it will suffice to show that $E_2 < \epsilon ws/6$. For this it will suffice to show that

$$3s^2(6\omega)^{2\ell x}\left[\frac{q+2\beta'}{1-\eta}\right]^{(1-\eta)\lfloor \ell x^*(u)\rfloor(\lfloor \ell x^*(v)\rfloor-1)} < \epsilon ws/6|F|^2$$

for all $uv \in E(F)$. We have $\lfloor \ell x^*(u)\rfloor\big(\lfloor \ell x^*(v)\rfloor - 1\big) > (1+\epsilon)\log_{1/q} n$, and so (since $\eta < \epsilon/8$) have $(1-\eta)\lfloor \ell x^*(u)\rfloor\big(\lfloor \ell x^*(v)\rfloor - 1\big) > (1+3\epsilon/4)\log_{1/q} n$. Now we have $\log\big((q+2\beta')/(1-\eta)\big) = \log(q+2\beta') + \log\big(1/(1-\eta)\big) = \log\big(1/(1-\eta)\big) - \log\big(1/(q+2\beta')\big) \leq \log\big(1/(1-\eta)\big) - (1-\epsilon/8)\log(1/q)$. Observe that $\log\big(1/(1-\eta)\big) = -\log(1-\eta) < 2\eta$ (since $\eta < \frac{1}{8}$), and $2\eta = \epsilon p/4$, and $q = 1-p < e^{-p}$ so $p < \log(1/q)$ whence $\log\big(1/(1-\eta)\big) < (\epsilon/4)\log(1/q)$. Thus $\log\big((q+2\beta')/(1-\eta)\big) < -(1-3\epsilon/8)\log(1/q)$. We now have

$$\begin{aligned}
&3s^2(6\omega)^{2\ell x}\left[\frac{q+2\beta'}{1-\eta}\right]^{(1-\eta)\lfloor \ell x^*(u)\rfloor(\lfloor \ell x^*(v)\rfloor-1)} \\
&\quad\leq\ s\exp\big[\log n + 2\ell x\log(1536|F|/\epsilon^2 wp) - (1+3\epsilon/4)(1-3\epsilon/8)\log n\big] \\
&\quad\leq\ s\exp\left[\Big(4x/\sqrt{m(\mathbf{x})}\Big)\sqrt{\log_{1/q} n}\,\log(1536|F|/\epsilon^2 wp) - (\epsilon/4)\log n\right] \\
&\quad\leq\ s\exp\left[\Big(4x/\sqrt{m(\mathbf{x})}\Big)\sqrt{(\log n)/p}\,\log(1536|F|/\epsilon^2 wp) - (\epsilon/4)\log n\right] \\
&\quad<\ \epsilon ws/6|F|^2
\end{aligned}$$

for n large, given the bounds on p. □

We now use this result to show that a dense graph G with reasonable connectivity has (to within a factor of $1-\epsilon$) as large an F_t minor as is to be found in a random graph. (The converse result for random graphs G, that no larger minor can be found, was found in Theorem 3.5; this result implies that minors of the given order can be found in random graphs, which almost surely have the required connectivity.) The corresponding result of Thomason [64] is his Theorem 4.1, where he shows that, given any s vertices of G, the K_s minor can be chosen with one of those given vertices in each part of the minor; this is used in the arguments for sparse graphs. Although the argument here would allow such vertices to be chosen to be in each part in our case as well, for the arguments of the next section it will suffice to use Thomason's original result where such a result is needed.

Theorem 3.12 *Let $0 < \epsilon < 1$. Let a weighted graph F be given. Then there exists N such that the following assertion holds.*

Let G be a graph of order $n > N$, edge density $p = 1 - q$ and connectivity $\kappa(G) \geq n(\log\log\log n)/(\log\log n)$. Write $q^* = \max\{q, (\log n)^{-1/\epsilon}\}$. Then $G \succ F_s$, where

$$s = \left\lceil (1-\epsilon)\frac{n\sqrt{m(F)}}{\sqrt{\log_{1/q} n}} \right\rceil .$$

Proof Assume throughout that n is large. By Lemma 3.4, for any u, $v \in V(G)$, u and v are joined in G by at least $\kappa^2/4n$ internally disjoint paths with length at most

$$h = 2(\log\log n)/(\log\log\log n);$$

let $P_{u,v}$ be the set of such paths.

Let $r = 1/(\log\log\log n)$ and select vertices independently and at random with probability r from $V(G)$, forming a set of vertices C, where $|C| < 2rn$ with probability at least $\frac{1}{2}$. Using Lemma 3.3, the probability that a given vertex $v \in G$ of degree $d(v)$ has more than $\epsilon d(v)/6$ neighbours within C is less than $1/n^2$. For given u, $v \in V(G)$, C contains all the internal vertices of some given path in $P_{u,v}$ with probability at least r^h, independently for each such path; and $r^h > (\log n)^{-1/6}$, so $r^h|P_{u,v}|/2 > n/(\log n)^{1/3}$. Again using

Lemma 3.3, we conclude that the probability that fewer than $r^h|P_{u,v}|/2$ paths of $P_{u,v}$ lie entirely within C is less than $1/n^3$; so there is some set C (which we now fix) with $|C| < 2rn$, with every vertex v of G having at most $\epsilon d(v)/6$ neighbours inside C, and, for every pair u, v of vertices of G, with at least $n/(\log n)^{1/3}$ internally disjoint paths from u to v with length at most h whose internal vertices lie within C.

Similarly, choose a random subset D of $V(G) \setminus C$, choosing each vertex with probability r. With probability at least $\frac{1}{2}$ we have $|D| < 2rn$; any given vertex v has at least $d(v)/2 \geq \kappa/2$ neighbours outside C and the probability that more than $\epsilon d(v)/6$ of these or fewer than $r\kappa/4$ of these lie in D is at most $1/n^2$; so we may fix D such that every vertex v has between $r\kappa/4$ and $\epsilon d(v)/6$ neighbours in D.

Let $\mathbf{x}$ be chosen so that $m(\mathbf{x}) = m(F)$. We will apply Corollary 3.11 to $G' = G - C - D$ to find an $(F_s, \mathbf{x})$-equipartition. The result will then immediately follow by Lemma 3.2, so it remains only to verify that the conditions of Corollary 3.11 do apply with suitable parameters. Let $n' = |G'|$ and let $p' = 1 - q'$ be the edge density of G'. We will use the parameters $\epsilon/8$, n' and p'. We have $n' \geq n(1-4r) = n\big(1-o(1)\big)$, and for all $v \in V(G')$ we have $d_{G'}(v) \geq (1-\epsilon/3)d_G(v)$. The connectivity of G implies that $p \geq \kappa/2n \geq (\log\log\log n)/(2\log\log n)$, so that $p' \geq (1-\epsilon/2)p > 1/(\log\log n)$. If $p' = o(1)$ we then have $\log(1/q') \approx p' \geq (1-\epsilon/2)p \approx (1-\epsilon/2)\log(1/q)$; we also have $q' \leq q\binom{n}{2}/\binom{n'}{2} \leq q/(1-8r)$ so that $\log(1/q') \geq \log(1/q) - 9r$ whence if $p' \neq o(1)$ we have $\log(1/q') = \big(1+o(1)\big)\log(1/q)$. In the case that q' is very small so that the upper bound on p in Corollary 3.11 does not hold, $q' < (\log n')^{-8/\epsilon}$, remove a few edges from G until this inequality no longer holds; whether or not we need to remove those edges, we now have $\log(1/q') \geq (1-3\epsilon/4)\log(1/q^*)$. Now the conditions of Corollary 3.11 hold;

applying it, we find an $(F_t, \mathbf{x})$-equipartition for some $t \geq s$, and so find an $(F_s, \mathbf{x})$-equipartition. □

3.5 Minors of sparse graphs

In this section we will show that $c(F_t) = \big(\alpha + o(1)\big)t\sqrt{(\log t)/m(F)}$. Consideration of random graphs with $q = \lambda$ and $n = t\sqrt{(\log_{1/\lambda} t)/m(F)}$ shows by Theorem 3.5 that $c(F_t)$ cannot be any smaller, so we need only show that graphs with the given average degree have the required minor. In the previous section we saw how large an F_t minor must be present in a dense reasonably connected graph; we will see that this is at least as large as is required. We will consider sparse graphs that are reasonably connected, and show that they have minors much larger than required, and then combine these arguments by considering graphs that are minor-minimal in certain classes. The minor-minimality will imply that every edge has many triangles on it, so vertex neighbourhoods are dense; so either there are many vertices whose neighbourhoods have much in common, or there are many whose neighbourhoods are largely disjoint. The former case is dealt with by the next lemma. The latter case is dealt with by finding many disjoint small complete minors in the disjoint neighbourhoods, and joining them up to form a large complete minor. These arguments are simply those of Thomason [64], but where the minors found are K_{Ct} for any constant C; he stated that the results were true for any constant C, but only proved them for $C = 2$. (The actual formulations given allow the constant in the average degree to be arbitrarily small, rather than the $\frac{3}{10}$ which was given in [64], and find a K_{2t} minor; this is clearly equivalent to the version where a larger minor is found.)

Lemma 3.13 *Let $0 < c < 1$ and positive integers f and g be given. Then there exist t_0 and $c_0 = c_0(c, f, g) > 0$ such that the following assertion holds.*

Let $t > t_0$ be a positive integer and let $d \geq ct\sqrt{\log t}$. Let G be a bipartite graph with vertex classes A and B with $|A| > c_0 d$ and $|B| < fd$ such that every vertex of A has degree at least d/g. Then $G \succ K_{2t}$.

(Thomason's version of the above lemma (for digraphs) has $c = \frac{3}{10}$, $f = 400$ and $g = 3$.)

Proof Let $c_0 = e^{100f^2g^2/c^2}$. We will show that $G \succ K_{2t}$, supposing throughout that t is large enough.

First remove edges as necessary so that each vertex $a \in A$ has degree exactly $\lceil d/g \rceil$. Now, successively for each vertex $a \in A$, select some neighbour b of a and contract the edge ab, until all the vertices of A have been identified with vertices of B and we have a graph on B only. Choose the neighbour b of a as the vertex of minimum degree in the subgraph spanned by the neighbours of a in the graph left at the stage when a is being dealt with. If this subgraph had edge density $p_a = 1 - q_a$ then contracting ab adds at least $q_a(\lceil d/g \rceil - 1)$ edges to the subgraph spanned by B. This graph has no more than $\frac{1}{2}|B|(|B| - 1)$ edges, so for some $a \in A$ we must have $q_a \leq q$ where $|A|q(\lceil d/g \rceil - 1) = |B|(|B| - 1)$, so $q = |B|(|B| - 1)/|A|(\lceil d/g \rceil - 1) < f^2d^2/c_0d(\lceil d/g \rceil - 1) < (f+1)^2g/c_0$. Let G' be the graph spanned by the neighbours of that vertex a when it is being dealt with, so that G' has density at least $1 - e^{-90f^2g^2/c^2}$. Note that $|G'| = \lceil d/g \rceil$.

We now find a subgraph of G' with high connectivity. Let S be the set of vertices of G' with degree less than $3|G'|/4$. We have $3|S||G'|/4 + |G' - S||G'| \geq (1 - q)|G'|(|G'| - 1)$ so $|S| < e^{-80f^2g^2/c^2}|G'|$. This means that $|G' - S| > d/(g + 1)$; and $\delta(G' - S) \geq 3|G'|/4 - |S|$ so $\kappa(G' - S) > |G'|/2$;

and $e(G'-S) \geq \frac{1}{2}(1-q)|G'|(|G'|-1)-|S||G'|$, so $G'-S$ has edge density at least $1-e^{-70f^2g^2/c^2}$. But now, by Theorem 4.1 of Thomason [64], $G'-S$ has a complete minor of order at least $\Big((1-\epsilon)c\big(1/(g+1)\big)\sqrt{70f^2g^2/c^2}\Big)\,t > 2t$, so G has our complete minor. □

We now show that in general a large sparse graph has a large complete minor.

Theorem 3.14 *Let* $0 < c < 1$ *be given. Then there exist* t_1 *and* c_1, $c_2 > 0$ *such that the following assertion holds.*

Let $t > t_1$ *be a positive integer and let* $d \geq ct\sqrt{\log t}$. *Let* G *be a graph with* $|G| \geq c_1 d$ *and* $\kappa(G) \geq c_2 t$. *Suppose that* $e(G) \leq d|G|$ *and that there are at least* d *triangles on every edge of* G. *Then* $G \succ K_{2t}$.

(Thomason's version of this theorem has $c = \frac{3}{10}$ and c_2 fixed to be 23.)

Proof Let $g = \lceil 4\alpha/c \rceil + 2$. Let $h = \binom{4g}{2}$. Let $f = 3h+1$. Let $c_1 = 3\big(c_0(c,f,3)+f\big)$ (where c_0 is the function of Lemma 3.13). Let $c_2 = h$. We will show that $G \succ K_{2t}$, supposing throughout that t is large enough.

We will first find $h+1$ disjoint subsets S_0, S_1, …, S_h of $V(G)$ such that all S_i satisfy $|S_i| \leq 3d$ and $\delta(G[S_i]) \geq 2d/3$. We successively find each S_{k+1} when all S_i have been found for $0 \leq i \leq k$. Write $B = \cup_{i=0}^{k} S_i$; we have $|B| \leq 3hd < fd$. Let $A = \{\, v \in V(G) \setminus B : d(v) \leq 3d \,\}$. We have $3d(|G|-|B|-|A|) \leq 2e(G) \leq 2d|G|$ so $|A| \geq |G|/3 - |B| \geq c_0 d$. Every edge of G is in at least d triangles, so for all $a \in A$ we have $\delta(G[\Gamma(a)]) \geq d$. If every vertex of A has at least $d/3$ edges to B, then by Lemma 3.13 we have our minor, so suppose there is some $a \in A$ with fewer than $d/3$ edges to B. Put $S_{k+1} = \Gamma(a) \setminus B$. Then S_{k+1} is disjoint from the previous S_i and we have $|S_{k+1}| \leq |\Gamma(a)| \leq 3d$ and $\delta(G[S_{k+1}]) \geq d - d/3 = 2d/3$. Thus we can find all the S_i with the required properties.

Now we find, for $0 \le i \le h$, a subset $T_i \subset S_i$ such that the average degree of $G[T_i]$ is at least $3d/5$ and $\kappa(G[T_i]) \ge d/40$. If $\kappa(G[S_i]) \ge d/40$ then take $T_i = S_i$; otherwise remove a cutset of minimum size and consider a smallest component of what remains; if that component has the required connectivity, take it as T_i, and otherwise keep repeating the process of removing a minimum cutset and taking a smallest component. After k steps this leaves a graph of order at most $2^{-k}|S_i| \le 3d2^{-k}$ and minimum degree at least $2d/3 - kd/40$. For $k = 3$ this is impossible so this process terminates after at most 2 steps, and then taking the resulting set as T_i we indeed have $\delta(G[T_i]) \ge 2d/3 - d/20 > 3d/5$, so we have the required connectivity and average degree.

Theorem 4.1 of Thomason [64] can be applied to the $G[T_i]$ to find complete minors in them. We will join the minors for $1 \le i \le h$ to form a larger minor using paths through T_0, and will need to find the paths first before finding the minors.

Let $s = 2\lceil t/2g \rceil$. For $1 \le i \le h$, take s distinct vertices $w_i^1, w_i^2, \ldots, w_i^s \in T_i$. By Menger's theorem [38] we then have hs entirely vertex-disjoint paths joining the set $W = \{ w_i^k : 1 \le k \le s, 1 \le i \le h \}$ to T_0; let $P(w)$ be the path joining $w \in W$ to T_0. The paths might use up many of the vertices of the T_i, so we need to adjust them not to do so. To make this adjustment, fix some $T = T_i$ for some $1 \le i \le h$. If $P(w)$ contains more than one vertex of T then let y_w and z_w be the first and last vertices of that path that are in T. Lemma 3.1 tells us that there are at least $10^{-5}d$ paths from y_w to z_w in T of length at most 240. But $10^{-5}d - 2hs - 240hs > hs$, so we may successively choose a subpath in T from y_w to z_w of length at most 240 for each w such that all those subpaths are vertex-disjoint. Thus we may adjust all the paths $P(w)$ such that no more than $240hs$ vertices of T_i lie on the paths for any $1 \le i \le h$. Let T'_i be the subset of T_i consisting of those vertices of T_i not in any of the paths $P(w)$.

We are now ready to find the minors within each T_i. Fix some $T = T_i$ with $1 \le i \le h$ and put $T' = T'_i$; then we have $|T| - 240hs \le |T'| \le 3d$ and $\kappa(G[T']) \ge d/40 - 240hs > d/41$, so Theorem 4.1 of Thomason [64] applies to $G[T']$. Write $n = |T'|$ and let ℓ be the average degree of $G[T']$ and $p = 1 - q$ be its edge density. Then $\log n = \big(1 + o(1)\big) \log t$ and $(n-1) = \ell/(1-q) \approx \ell/(1-q^*)$, so $G[T']$ has a K_r minor with $r \ge (1-\epsilon)\ell(1-q^*)^{-1}\sqrt{\log(1/q^*)}\big/\sqrt{\log t}$ (for any $\epsilon > 0$, provided t is sufficiently large). The expression in q^* is minimal at $q^* = \lambda$, where $(1-q^*)^{-1}\sqrt{\log 1/q^*} = 1/2\alpha$, and $\ell \ge 3d/5 - 240hs > d/2$, so $r > ct/4\alpha > 2\lceil t/2g \rceil = s$. Thus $G[T']$ has a K_s minor in which each of $w_i^1, w_i^2, \ldots, w_i^s$ is in a separate part of the minor.

It remains to join up these K_s minors to form a K_{2t} minor. Relabel the $h = \binom{4g}{2}$ sets T_i as $T_{i,j}$ for $1 \le i < j \le 4g$. Put $m = \lceil t/g \rceil = s/2$. Relabel the set $W \cap T_{i,j}$ as $\{\, a_{i,j}^k, b_{i,j}^k : 1 \le k \le m \,\}$. We saw that $T_{i,j}$ has a K_{2m} minor, whose parts we may identify as $A_{i,j}^k$ and $B_{i,j}^k$ for $1 \le k \le m$ with $a_{i,j}^k \in A_{i,j}^k$ and $b_{i,j}^k \in B_{i,j}^k$ for all k.

We now choose $4gm \ge 2t$ sets U_i^k, for $1 \le i \le 4g$ and $1 \le k \le m$, which will be extended to form our minor. Define

$$U_i^k = \Big(\bigcup_{j<i} A_{j,i}^k\Big) \cup \Big(\bigcup_{j<i} V\left(P(a_{j,i}^k)\right)\Big) \cup \Big(\bigcup_{j>i} B_{i,j}^k\Big) \cup \Big(\bigcup_{j>i} V\left(P(b_{i,j}^k)\right)\Big).$$

These sets are manifestly disjoint. There is an edge between distinct sets U_i^k and U_j^l: between $B_{i,j}^k$ and $A_{i,j}^l$ if $i < j$, or between $A_{i-1,i}^k$ and $A_{i-1,i}^l$ or between $B_{i,i+1}^k$ and $B_{i,i+1}^l$ if $i = j$ but $k \ne l$. Thus it remains only to make them connected. Each U_i^k contains exactly $4g-1$ vertices of T_0, the endpoints of the paths $P(a_{j,i}^k)$ for $j < i$ and $P(b_{i,j}^k)$ for $j > i$; each $A_{j,i}^k \cup V\left(P(a_{j,i}^k)\right)$ spans a connected subgraph, as does each $B_{i,j}^k \cup V\left(P(b_{i,j}^k)\right)$, so $G[U_i^k]$ has at most $4g-1$ connected components, each containing a vertex of T_0. Thus we need only join up these components with $4g-2$ internally disjoint paths in T_0, calling the set U_i^k with these paths added W_i^k, making these sets of paths for all i and k vertex-disjoint from each other. We need to choose a total of $4g(4g-2)m < hs$ paths, and just as we found short disjoint paths in T_i for $1 \leq i \leq h$ we can find the required number of short disjoint paths in T_0. Thus we have found our minor. $\square$

To deal with graphs with small connectivity, we now need a notion of minor-minimality, which is defined in terms of classes $\mathcal{G}_{d,k}$ of graphs, analogous to the classes $\mathcal{D}_{d,k}$ of digraphs considered by Thomason [64]. For $d \in \mathbb{N}$ a positive integer and $k \leq (d+1)/2$ with $2k \in \mathbb{N}$, put

$$\mathcal{G}_{d,k} = \{\, G : |G| \geq d \text{ and } e(G) > d|G| - kd \,\}.$$

Say that a graph G is *minor-minimal in* $\mathcal{G}_{d,k}$ if $G \in \mathcal{G}_{d,k}$ but, for all $H \prec G$, if $H \neq G$ then $H \notin \mathcal{G}_{d,k}$. Let G be a minor-minimal graph in $\mathcal{G}_{d,k}$. Observe that $K_d \notin \mathcal{G}_{d,k}$ and $K_{d+1} \notin \mathcal{G}_{d,k}$, so that $|G| \geq d+2$. Considering removing an edge shows that $e(G) = d|G| - kd + 1$; considering removing a vertex shows that $d+1 \leq \delta(G)$; we have $\delta(G) \leq 2d-1$ from the value of $e(G)$; and considering contracting an edge shows that at least d triangles lie on every edge of G. Finally, we claim that $\kappa(G) > k$. To see this, let S be a cutset and C a component of $G-S$. Then $G[C \cup S]$ and $G-C$ are minors of G with more than d vertices (by the minimum degree). Minor-minimality means that they are not in $\mathcal{G}_{d,k}$, so that $e(G[C \cup S]) \leq d|C| + d|S| - kd$ and $e(G-C) \leq d|G| - d|C| - kd$, whence $e(G) \leq d|G| + d|S| - 2kd$. But since $e(G) > d|G| - kd$, we have $d|S| > kd$, so $\kappa(G) > k$.

We are now ready to prove the general extremal result for blown-up graphs.

Theorem 3.15 *Let F be a weighted graph. Let $\alpha = 0.3190863431\dots$ be the constant defined above. Let $c(F_t)$ be the function defined above. Then*

$$c(F_t) = \big(\alpha + o(1)\big)t\sqrt{(\log t)/m(F)}.$$

Proof As noted above, $c(F_t)$ cannot be any smaller than stated, by Theorem 3.5. Take some $\epsilon > 0$, and let $d = \left\lceil(\alpha+\epsilon)t\sqrt{(\log t)/m(F)}\right\rceil$. It will then suffice to prove that, if t is sufficiently large (in terms of ϵ), then any graph G with $e(G) \geq d|G|$ has an F_t minor.

Put $k = \lceil d/\log\log\log d\rceil$. Then $e(G) \geq d|G|$ implies that $G \in \mathcal{G}_{d,k}$. Thus it suffices to show that, for t sufficiently large, if G is minor-minimal in $\mathcal{G}_{d,k}$ then $F_t \prec G$. So now suppose that G is minor-minimal in $\mathcal{G}_{d,k}$.

We now have $e(G) = d|G| - kd + 1 = d|G|\big(1+o(1)\big)$, at least d triangles on every edge of G, and $\kappa(G) > k$. If c_1 is the constant of Theorem 3.14 applied with $c = \alpha/2\sqrt{m(F)}$, then if $|G| \geq c_1 d$, then provided t is sufficiently large the connectivity condition of Theorem 3.14 also applies and $K_{2t} \prec G$, so $F_t \prec G$. So we now suppose that $|G| \leq c_1 d$. Put $n = |G|$ and let $p = 1-q$ be the edge density of G. Again provided t is sufficiently large, the conditions of Theorem 3.12 now apply. We have $\log n = \big(1+o(1)\big)\log t$, and if ℓ is the average degree of G we have $(n-1) = \ell/(1-q) \approx \ell/(1-q^*)$, so that we have an F_s minor with $s \geq (1-\epsilon/2)\ell(1-q^*)^{-1}\sqrt{\log(1/q^*)}/\sqrt{\log t}$. Since $\ell = 2d\big(1+o(1)\big)$, we have an F_s minor with $s \geq 2\alpha t(1-q^*)^{-1}\sqrt{\log(1/q^*)}$. The expression in q^* is minimal at $q^* = \lambda$, where $(1-q^*)^{-1}\sqrt{\log(1/q^*)} = 1/2\alpha$, so indeed we have an F_s minor with $s \geq t$. □

Chapter 4

Sparse bipartite minors

4.1 Introduction

In Chapter 2 we saw when complete bipartite minors appear in random graphs. Where the two parts of the bipartite graph have sizes in some constant ratio, and we ask for how large a $K_{\beta t,(1-\beta)t}$ minor appears in a random graph, we saw that $t = n\Big/\sqrt{4\beta(1-\beta)\log_{1/q} n}$. In Chapter 3, we saw what average degree forces such a minor in a general graph, seeing that the extremal graphs are derived from random graphs with a certain order and density. (The details of the form of the extremal graphs, where the excluded minor is a complete graph, are derived in Section 5.5.)

As well as considering the case where the size of the parts of the complete bipartite graph are in constant ratio, it is also natural to consider the extreme cases, of $K_{s,t}$ where s is fixed and t is large. In this case, the extremal graphs are no longer random. For example, in the trivial case where the graph we wish to avoid as a minor is the star $K_{1,t}$, the extremal graphs avoiding this minor are the union of disjoint K_t graphs.

In many cases, it seems that $K_s + \overline{K_t}$ minors occur just when $K_{s,t}$ minors do; in Chapter 2 we saw this to be the case for minors in random graphs. For this reason, we also discuss $K_s + \overline{K_t}$ minors in this chapter, although without proving results for them.

Of course, an average degree $O\left(t\sqrt{\log t}\right)$ forces a K_{s+t} minor, and so a $K_{s,t}$ minor. However, a better bound on the average degree that forces such a minor would be desirable. I make the following conjecture.

Conjecture 4.1 *Let s be a positive integer. Then there exists a constant C such that, for all positive integers t, if G has average degree at least Ct, then $K_{s,t} \prec G$.*

In this chapter we determine the exact average degree that forces a $K_{2,t}$ minor, for t sufficiently large. The results are only stated and proved for $K_{2,t}$ minors, but many of the arguments are more general and indications are given where appropriate of how they may be adapted to $K_s + \overline{K_t}$ minors. These indications only describe how the arguments given could be generalised; in some places, additional arguments not given here would also be needed. The arguments for $K_{2,t}$ minors are much more complicated than one might expect; this complexity seems necessary, although much of it is only needed to achieve a best possible average degree of $t + 1$; an average degree of $t + 2$ can be achieved without many of the special cases; in particular, none of the special cases in Lemma 4.8 are needed for such a weaker result. We return to $K_{s,t}$ and $K_s + \overline{K_t}$ minors at the end of the chapter.

4.2 Simple bounds

We observed that the star is a trivial case of a complete bipartite minor. We state the obvious bounds for star minors formally here.

Theorem 4.2 *Let $t \geq 1$ be some integer. If a graph has average degree greater than $t-1$ then it has a $K_{1,t}$ minor, but there exist arbitrarily large graphs with average degree $t-1$ and no $K_{1,t}$ minor.*

Proof For the first part, if a graph has average degree greater than $t-1$ then it has a vertex v of degree at least t, and v together with t of its neighbours provides a $K_{1,t}$ subgraph, which is a minor. For the second part, consider graphs that are the union of arbitrarily many disjoint K_t subgraphs. □

The following construction provides a general lower bound, which turns out to be the correct bound for $s = 2$.

Theorem 4.3 *Let $2 \leq s \leq t$ be integers and let $k \geq 1$ be an integer. Let G be the graph on $kt+s-1$ vertices that is the union of k graphs K_{t+s-1}, there being $s-1$ vertices shared among all those K_{t+s-1} and all the other vertices of G being in exactly one K_{t+s-1}. Then G does not contain a $K_{s,t}$ minor.*

Proof Suppose that G has a $K_{s,t}$ minor, so that there are disjoint subsets $V_1, V_2, \ldots, V_s, W_1, W_2, \ldots, W_t$ of $V(G)$ such that all $G[V_i]$ and $G[W_j]$ are connected and there is an edge from V_i to W_j for all i and j.

Because there are only $s-1$ vertices of G shared among all the K_{t+s-1}, at least one of the V_i does not contain any of those vertices; likewise, since $s \leq t$, at least one of the W_j does not contain any of those vertices. There must be an edge between any such V_i and W_j, so all such V_i and W_j lie entirely within the same t vertices that are in just one of the K_{t+s-1} making up G. All other V_i and W_j must have a vertex in the $s-1$ shared vertices; but this implies that all V_i and W_j have at least one vertex within the same K_{t+s-1}, a contradiction since the V_i and W_j are disjoint and there are $t+s$ of them.

□

Corollary 4.4 *Let $2 \leq s \leq t$ be integers. For any $\epsilon > 0$, there exist arbitrarily large graphs G with average degree at least $t + 2s - 3 - \epsilon$ and no $K_{s,t}$ minor.*

Proof The graph G of Theorem 4.3 has

$$k\left(\tfrac{1}{2}t(t-1) + t(s-1)\right) + \tfrac{1}{2}(s-1)(s-2) = k\left(\tfrac{1}{2}t(t+2s-3)\right) + \tfrac{1}{2}(s-1)(s-2)$$

edges. This gives an average degree of

$$\frac{kt(t+2s-3) + (s-1)(s-2)}{kt+s-1} = t + 2s - 3 - \frac{(s-1)(t+s-1)}{kt+s-1},$$

which tends to $t + 2s - 3$ from below as $k \to \infty$. □

4.3 Small graphs

In this section we show that a graph of order not much bigger than t, and with more than $\frac{t+1}{2}(|G| - 1)$ edges, has a $K_{2,t}$ minor.

In general we consider a graph G, of order $t + d$, where $d < \frac{1}{11}t^{1/4}$, and suppose that this graph has more than $\frac{t+1}{2}(|G| - 1)$ edges. Clearly we need only consider $d \geq 2$. We then show that there are disjoint subsets A and B of $V(G)$, such that $G[A]$ and $G[B]$ are connected, A has at least $t + \frac{d}{2}$ neighbours outside A, and B has at least $t + \frac{d}{2}$ neighbours outside B. Then A and B provide one half of the minor, and the intersection of the sets of neighbours provides the other half. Each of A and B will in fact consist of a single vertex, or a pair of neighbouring vertices.

The case of $d \leq 3$ turns out to be a special case, which we readily dispose of:

Lemma 4.5 *Let t be a positive integer. Let G be a graph of order $t + 2$ or $t + 3$ with more than $\frac{t+1}{2}(|G| - 1)$ edges. Then G has a $K_{2,t}$ minor.*

Proof If G is of order $t+2$, then it has two vertices of degree $t+1$; for otherwise, $e(G) \leq \frac{1}{2}(t|G|+1) = \frac{1}{2}(t^2+2t+1) = \frac{t+1}{2}(|G|-1)$, a contradiction. Those two vertices have t common neighbours, yielding our minor.

Now suppose G is of order $t+3$, so it has at least $\frac{1}{2}(t+1)(t+2)+1 = \frac{1}{2}(t^2+3t+4)$ edges. If G has a vertex x of degree $t+2$, then it has some other vertex y of degree at least $t+1$; for otherwise, $e(G) \leq \frac{1}{2}(t|G|+2) = \frac{1}{2}(t^2+3t+2)$, a contradiction. Then x and y have t common neighbours. Otherwise, we see in the same way that G must have at least four vertices of degree $t+1$. If any two of these are nonneighbours, then they have $t+1$ common neighbours. Suppose then that there are exactly k vertices of degree $t+1$ and none of greater degree; let those vertices be x_1, x_2, $\ldots$, x_k. Each of them has $t+2-k$ neighbours in the rest of the graph, so exactly one nonneighbour in the rest of the graph; let the nonneighbour of x_i be y_i. No two y_i are the same, since if $y_i = y_j$ then x_i and x_j would have t common neighbours. If there were an edge between y_i and y_j then contracting that edge would yield our minor, one half having the vertices x_i and x_j and the other half having all the other vertices of the new graph. Thus there are no edges among the y_i. The degrees of all vertices of G add up to at least $t|G|+4$; all vertices other than the x_i have degree at most t, so we must have $\sum_{v\in A} d(v) \geq t|A|-k+4$ for any $A \subset V(G)\setminus\{x_1,x_2,\ldots,x_k\}$. But we have $d(y_i) \leq t+2-k$ for $1 \leq i \leq k$, and since $k \geq 4$ this yields a contradiction by taking $A = \{y_1, y_2, \ldots, y_k\}$. $\square$

For larger d, we find A and B separately, finding both of them by the same method; our results will show that, given a set $X \subset V(G)$ with $|X| \leq 2$, there is a subset $Y \subset V(G)\setminus X$ with $G[Y]$ connected, $|Y| \leq 2$ and Y having at least $t+\frac{d}{2}$ neighbours outside Y. This can then be applied with X empty, to find A, then with $X = A$, to find B.

Lemma 4.6 *Let t and d be positive integers. Let G be a graph of order $t+d$. Let $X \subset V(G)$ with $|X| \leq 2$. Let the maximum degree (in G) of any vertex in $V(G) \setminus X$ be $t+s$, where $0 \leq s < \frac{d}{2}$. Let j be an integer with $0 \leq j \leq s$, and write $d_h = \lceil \frac{d}{2} - j \rceil$. Suppose that there does not exist a subset $Y \subset V(G) \setminus X$ with $G[Y]$ connected, $|Y| \leq 2$ and Y having at least $t + \frac{d}{2}$ neighbours outside Y. Then G has at most $\frac{1}{2}(t+d-1)(t+1) + \frac{1}{2}[t(d_h + s + j - d + 1) + d^2 + j^2 - 2dj - 1 + jd_h + js - 2d_h]$ edges.*

Proof Let v be some vertex not in X with degree at least $t+j$. Let A be a set of $t+j$ neighbours of v, and let B be the set of the remaining $d-j-1$ vertices of G.

Within B, there are at most $\binom{d-j-1}{2} = \frac{1}{2}(d^2 + j^2 - 2dj - 3d + 3j + 2)$ edges. From v to the rest of the graph there are at most $t+s$ edges. It remains to maximise the number of edges within A, plus the number from A to B. If we write $d_A(x)$ for the number of edges within A from a vertex $x \in A$, and $d_B(x)$ for the number of edges to B, then we need to maximise $\sum_{x \in A} d_B(x) + \frac{1}{2} d_A(x)$. Every vertex x in $A \setminus X$ has $d_B(x) \leq d_h$ (since if $x \in A \setminus X$ had $\frac{d}{2} - j + 1$ neighbours in B, we could take $Y = \{v, x\}$). Every vertex x in $A \setminus X$ has $d_A(x) + d_B(x) \leq t+s-1$. To maximise $d_B(x) + \frac{1}{2} d_A(x)$ subject to these constraints, we have $d_B(x) = d_h$ and $d_A(x) = t+s-1-d_h$, so we deduce $d_B(x) + \frac{1}{2} d_A(x) \leq d_h + \frac{1}{2}(t+s-1-d_h)$, for vertices $x \in A \setminus X$. For any vertices $x \in A \cap X$, we know only that $d_B(x) + \frac{1}{2} d_A(x) \leq |B| + \frac{1}{2}(|A|-1) = d - j - 1 + \frac{1}{2}(t+j-1)$. Note that $\sum_{x \in A} d_B(x) + \frac{1}{2} d_A(x)$ will be maximised if $|A \cap X| = 2$.

Thus, we have

$$\begin{aligned} e(G) \leq\ & \tfrac{1}{2}(d^2 + j^2 - 2dj - 3d + 3j + 2) + (t+s) \\ & + (t+j-2)\left(d_h + \tfrac{1}{2}(t+s-1-d_h)\right) \end{aligned}$$

$$
\begin{aligned}
&\quad + 2\big(d - j - 1 + \tfrac{1}{2}(t + j - 1)\big) \\
&= \tfrac{1}{2}[d^2 + j^2 - 2dj - 3d + 3j + 2 + (2t + 2s) \\
&\qquad + (t + j - 2)(d_h + t + s - 1) + (4d - 2j - 6 + 2t)] \\
&= \tfrac{1}{2}[d^2 + j^2 - 2dj + d + j - 4 + 4t + 2s \\
&\qquad + (t + j - 2)(d_h + t + s - 1)] \\
&= \tfrac{1}{2}\big[d^2 + j^2 - 2dj + d + j - 4 + 4t + 2s \\
&\qquad + \big(t^2 + t(d_h + s + j - 3) + (j - 2)(d_h + s - 1)\big)\big] \\
&= \tfrac{1}{2}[t^2 + t(d_h + s + j + 1) \\
&\qquad + d^2 + j^2 - 2dj + d - 2 + jd_h + js - 2d_h] \\
&= \tfrac{1}{2}(t + d - 1)(t + 1) \\
&\quad + \tfrac{1}{2}[t(d_h + s + j - d + 1) + d^2 + j^2 - 2dj - 1 + jd_h + js - 2d_h].
\end{aligned}
$$

□

Corollary 4.7 *Let t and d be positive integers, with $4 \le d < \sqrt{t}$. Let G be a graph of order $t + d$ with more than $\frac{t+1}{2}(t + d - 1)$ edges. Let $X \subset V(G)$ with $|X| \le 2$. Let the maximum degree (in G) of any vertex in $V(G) \setminus X$ be $t + s$. Then, if $s > \frac{d-1}{2}$ or $s < \frac{d-3}{2}$, $V(G)$ has a subset $Y \subset V(G) \setminus X$ with $G[Y]$ connected, $|Y| \le 2$ and Y having at least $t + \frac{d}{2}$ neighbours outside Y.*

Proof If $s > \frac{d-1}{2}$, then $s \ge \frac{d}{2}$, and Y can be a single vertex with degree $t+s$. If $s < 0$, we would have $e(G) \le \frac{1}{2}\big(t(|G| - 1) + 2(d - 1)\big) = \frac{1}{2}\big(t(t + d - 1) + 2d - 2\big) = \frac{1}{2}\big((t + 1)(t + d - 1) + d - 1 - t\big)$, a contradiction. Thus we have $0 \le s < \frac{d-3}{2}$.

Suppose for a contradiction that there is no such Y. Put $j = 1$ in Lemma 4.6. If d is even, we have $s \le \frac{d}{2} - 2$ and $d_h = \frac{d}{2} - 1$; if d is odd, we have $s \le \frac{d}{2} - \frac{5}{2}$ and $d_h = \frac{d}{2} - \frac{1}{2}$. In either case, $d_h + s \le d - 3$. We then have

$\frac{1}{2}(t+d-1)(t+1)+1 \le e(G) \le \frac{1}{2}(t+d-1)(t+1)+\frac{1}{2}[t(d_h+s+2-d)+d^2-2d+s-d_h]$. We deduce that

$$\begin{aligned} 0 &\le t(d_h+s+2-d)+d^2-2d-2+s-d_h \\ &\le -t+d^2, \end{aligned}$$

a contradiction by the constraint on the value of d. □

Lemma 4.8 *Let t and d be positive integers, with $4 \le d < \frac{1}{11}t^{1/4}$. Let G be a graph of order $t+d$ with more than $\frac{t+1}{2}(|G|-1)$ edges. Let $X \subset V(G)$ with $|X| \le 2$. Let the maximum degree (in G) of any vertex in $V(G) \setminus X$ be $t+s$, where $\frac{d-3}{2} \le s \le \frac{d-1}{2}$. Then either G has a $K_{2,t}$ minor or $V(G)$ has a subset $Y \subset V(G) \setminus X$ with $G[Y]$ connected, $|Y| \le 2$ and Y having at least $t+\frac{d}{2}$ neighbours outside Y.*

Proof We work as in the proof of Lemma 4.6, taking $j=s$. Let v, A and B be as in that proof. If $s=\frac{d-3}{2}$, we have $d_h=2$; otherwise we have $d_h=1$. Note that, while $d_B(x)+\frac{1}{2}d_A(x)$ is maximised if $d_B(x)=d_h$ and $d_A(x)=t+s-1-d_h$, if s is too large then $d_A(x) \ge t$ and v and x have t common neighbours, giving a $K_{2,t}$ minor. Write $c=2s-(d-3)$, so $d=2s+3-c$. Say a vertex $x \in A \setminus X$ is *good* if $d_B(x)+\frac{1}{2}d_A(x) \le \frac{1}{2}(t+s-c)$, *poor* if $d_B(x)+\frac{1}{2}d_A(x)=\frac{1}{2}(t+s-c+1)$ and *bad* if $d_B(x)+\frac{1}{2}d_A(x) > \frac{1}{2}(t+s-c+1)$.

If $s=\frac{d-1}{2}$, we have $c=2$. Considering the two possible values for $d_B(x)$, we see that if $s \ge 3$ there can be no bad or poor vertices, but if $s=2$ (so $d=5$) there can be no bad vertices but there can be poor vertices with $d_B(x)=1$ and $d_A(x)=t+s-3=t-1$. If $s=\frac{d-2}{2}$, we have $c=1$; if $s \ge 2$ there can be no bad or poor vertices, but if $s=1$ (so $d=4$) there can be no bad vertices but there can be poor vertices with $d_B(x)=1$ and $d_A(x)=t+s-2=t-1$. Finally, if $s=\frac{d-3}{2}$, and so $c=0$, again there can be no bad vertices, and if $s \geq 3$ there can be no poor vertices, but if $s = 2$ (so $d = 7$) there can be poor vertices with $d_B(x) = 2$ and $d_A(x) = t + s - 3 = t - 1$ and if $s = 1$ (so $d = 5$) there can be poor vertices with $d_B(x) = 2$ and $d_A(x) = t + s - 3 = t - 2$.

First suppose that there are at least $\frac{1}{60}\sqrt{t}$ good vertices; this will hold in particular when all vertices are good, which always occurs except in the four cases given above when there may be poor vertices. Supposing there is no $K_{2,t}$ minor, and no Y with the property of the lemma, we maximise the number of edges in the graph. The number of edges within B is at most $\binom{d-s-1}{2} = \binom{s+2-c}{2}$; the number from v to the rest of the graph is $t + s$; and $|B| = d - s - 1 = s + 2 - c$; so we have

$$\begin{aligned}
e(G) \ &\leq\ \binom{s+2-c}{2} + (t+s) + \tfrac{1}{2}(t+s-2)(t+s-c+1) - \tfrac{1}{2}\left(\tfrac{1}{60}\sqrt{t}\right) \\
&\qquad + 2\left(s+2-c+\tfrac{1}{2}(t+s-1)\right) \\
&=\ \tfrac{1}{2}[(s+2-c)(s+1-c) + (2t+2s) + (t+s-2)(t+s-c+1) \\
&\qquad - \tfrac{1}{60}\sqrt{t} + (2t+6s+6-4c)] \\
&=\ \tfrac{1}{2}[t^2 + dt - \tfrac{1}{60}\sqrt{t} + (s+2-c)(s+1-c) \\
&\qquad + 8s + (s-2)(s-c+1) + 6 - 4c] \\
&=\ \tfrac{1}{2}(t+1)(t+d-1) \\
&\qquad + \tfrac{1}{2}[(c-2-2s) - \tfrac{1}{60}\sqrt{t} + (s^2+3s-2cs+2+c^2-3c) \\
&\qquad\quad + 8s + (s^2-s-cs-2+2c) + 6 - 4c] \\
&=\ \tfrac{1}{2}(t+1)(t+d-1) + \tfrac{1}{2}[-\tfrac{1}{60}\sqrt{t} + 2s^2 + 8s - 3cs + c^2 - 4c + 4].
\end{aligned}$$

Since $e(G) > \frac{1}{2}(t+1)(t+d-1)$, we have

$$\begin{aligned}
\tfrac{1}{60}\sqrt{t} &< 2s^2 + 8s - 3cs + c^2 - 4c + 4 \\
&< d^2/2 + d/4 + 8
\end{aligned}$$

$$< \ 2d^2,$$

so $\sqrt{t} < 120d^2$. But since $d < \frac{1}{11}t^{1/4}$ we have $\sqrt{t} > 121d^2$, a contradiction.

It remains to consider the case where there are fewer than $\frac{1}{60}\sqrt{t}$ good vertices. Note that we have $t > 44^4 = 3748096$. Let A_G be the subset of A consisting of all vertices that are either good or in X; we then have $|A_G| \leq \frac{1}{60}\sqrt{t} + 2 < \frac{1}{56}\sqrt{t}$. All other vertices of A are poor.

In each of the four cases enumerated above where there can be poor vertices, the poor vertices all have the same d_A and d_B values, and B is of a small constant size (2, 3 or 4 depending on the case). The cases are as follows:

- $d = 4$, $|A| = t + 1$, $d_A = t - 1$, $|B| = 2$, $d_B = 1$.
- $d = 5$, $|A| = t + 2$, $d_A = t - 1$, $|B| = 2$, $d_B = 1$.
- $d = 5$, $|A| = t + 1$, $d_A = t - 2$, $|B| = 3$, $d_B = 2$.
- $d = 7$, $|A| = t + 2$, $d_A = t - 1$, $|B| = 4$, $d_B = 2$.

In all these cases, $d_A + d_B \geq t$. We find a minor in one of two ways. First, if poor vertices x, $y \in A$ have $\Gamma_B(x) = \Gamma_B(y)$, then let one part of the minor be y and another be $\{v, x\}$. If x and y are not neighbours then those parts of the minor have at least t common neighbours; in the case where $d = 7$ and $d_A + d_B = t + 1$, those parts have t common neighbours even if x and y are neighbours. Second, we try to find poor vertices x_1, y_1, x_2, $y_2 \in A$ such that $\Gamma_B(x_1) = \Gamma_B(y_1)$ and $\Gamma_B(x_2) = \Gamma_B(y_2)$, such that x_1 and x_2 are neighbours, and y_1 and y_2 are neighbours, and $\{x_1, x_2\}$ and $\{y_1, y_2\}$ have t common neighbours so may be taken as the parts of our minor.

The simplest case to consider is that of $d = 7$. Here we only need two poor vertices with the same neighbours in B. We will have these as long as we have at least $7 = \binom{4}{2} + 1$ poor vertices, which we do by the bounds on t and $|A_G|$.

The next simplest case to consider is that of $d = 4$. Here each poor vertex has exactly one nonneighbour (which may or may not be poor) in A. By the above arguments, we may suppose that any two poor vertices that are not neighbours have different neighbours in B. If two poor vertices x and y are neighbours but share the same neighbour in B and the same nonneighbour in A, then they have $t-2$ common neighbours in A, one common neighbour in B, and share the neighbour v, so we have our minor. Thus we may suppose that any element of A_G is a nonneighbour of at most two poor vertices. Thus there are at least 4 poor vertices which have poor nonneighbours, and so the poor vertices include at least 2 pairs of nonneighbours. Say that x_1 and y_2 are nonneighbours, and x_2 and y_1 are nonneighbours, where $\Gamma_B(x_1) = \Gamma_B(y_1) = \{b_1\}$, say, and $\Gamma_B(x_2) = \Gamma_B(y_2) = \{b_2\}$. Then $\{x_1, x_2\}$ and $\{y_1, y_2\}$ each have as neighbours the t other vertices of the graph, and we have our minor.

Now consider the case where $d = 5$ and $|B| = 2$. Let $B = \{b_1, b_2\}$ and write A_1 for the set of those poor vertices whose neighbour in B is b_1, and A_2 for the set of those poor vertices whose neighbour in B is b_2. Each poor vertex has exactly 2 nonneighbours in A; by the above arguments, all edges within A_1 are present, as are all edges within A_2. We will find x_1, $y_1 \in A_1$ and x_2, $y_2 \in A_2$ such that x_1 and x_2 are neighbours; y_1 and y_2 are neighbours; $\{x_1, x_2\}$ has as neighbours all but at most one vertex; $\{y_1, y_2\}$ has as neighbours all but at most one vertex; and, if both those sets do not have as neighbours all of G, their nonneighbours (which can only be in A_G) are the same. This will yield our minor.

To find those vertices, first observe that there can be no more than 2 poor vertices with any given pair of nonneighbours in A_G (since two such with the same neighbour in B would have t common neighbours). Thus there are fewer than $|A_G|^2$ poor vertices with both nonneighbours in A_G. Let A'_1 be the result of removing all such vertices from A_1, and let A'_2 be the result of removing all such vertices from A_2. At least one of these sets has order at least $5|A_G|$; without loss of generality suppose that is A'_1. Then there are at least 5 vertices in A'_1 that, if they have any nonneighbour in A_G, have the same nonneighbour in A_G. Let those be A''_1.

Now take any vertex $x_1 \in A''_1$, and let y_2 be a nonneighbour of x_1 in A_2. If x_1 has any other nonneighbour z in A_2, remove from A''_1 all nonneighbours (at most 2) of z. Also remove from A''_1 any nonneighbour (other than x_1) of y_2. There is at least one vertex other than x_1 left in A''_1; let y_1 be such a vertex. Let x_2 be a nonneighbour in A_2 of y_1. Then x_1 and x_2 are neighbours, as are y_1 and y_2, and each pair has as neighbours v, all of B, all of A_1 and A_2, and all of A_G except possibly the single vertex allowed to be a nonneighbour of vertices in A''_1. Thus we have our minor.

Finally, consider the case where $d = 5$ and $|B| = 3$. Let $B = \{b_1, b_2, b_3\}$ and write A_{12} for the set of those poor vertices whose neighbours in B are $\{b_1, b_2\}$, and define A_{23} and A_{31} likewise. Each poor vertex has exactly 2 nonneighbours in A; by the above arguments, all edges within A_{12} are present, as are all edges within A_{23} and all edges within A_{31}. For some pair of those sets—say A_{12} and A_{23}—we will find $x_{12}, y_{12} \in A_{12}$ and $x_{23}, y_{23} \in A_{23}$ such that x_{12} and x_{23} are neighbours; y_{12} and y_{23} are neighbours; $\{x_{12}, x_{23}\}$ has as neighbours all but at most one vertex; $\{y_{12}, y_{23}\}$ has as neighbours all but at most one vertex; and, if both those sets do not have as neighbours all of G, their nonneighbours (which can only be in $A_G \cup A_{31}$) are the same. This will yield our minor.

To find those vertices, first observe that there can be no more than 3 poor vertices with any given pair of nonneighbours in A_G. Thus there are fewer than $\frac{3}{2}|A_G|^2$ poor vertices with both nonneighbours in A_G. Let A'_{12}, A'_{23} and A'_{31} be the result of removing all such vertices from A_{12}, A_{23} and A_{31} respectively. There are at least $48|A_G|$ vertices left after this removal, so some one of those sets, without loss of generality A'_{12}, has at least $16|A_G|$ vertices. Each vertex of A'_{12} has a nonneighbour in A_{23} or A_{31}, so without loss of generality suppose that at least $8|A_G|$ vertices have a nonneighbour in A_{23}, letting the set of such vertices be A''_{12}. Dividing up those vertices according to what nonneighbour, if any, they have in A_G, we arrive at a subset A'''_{12} with at least 8 vertices all of which have the same nonneighbour, if any, in A_G.

Now let x_{12} be any vertex of A'''_{12}, and let y_{23} be a nonneighbour of x_{12} in A_{23}. Remove from A'''_{12} the following vertices: any nonneighbour (other than x_{12}) of y_{23} (at most 1 vertex); any nonneighbours (other than x_{12}) of any nonneighbour (other than y_{23}) of x_{12} in A_{23} (at most 2 vertices); any vertex in A_{12} that shares a nonneighbour in A_{31} with y_{23} (at most 1 vertex); any nonneighbours in A_{12} of any vertex in A_{23} that shares a nonneighbour in A_{31} with x_{12} (at most 2 vertices). At least one vertex other than x_{12} remains in A'''_{12}. Let y_{12} be such a vertex, and let x_{23} be a nonneighbour of y_{12} in A_{23}. Then x_{12} and x_{23} are neighbours; y_{12} and y_{23} are neighbours; and each pair has as neighbours v, all of B, all of A_{12}, all of A_{23}, all of A_{31} (since we arranged that neither pair could share a nonneighbour in A_{31}), and all of A_G except possibly the one vertex allowed to be a nonneighbour of vertices in A'''_{12}. Thus we have our minor. □

Given these results, we can now conclude that a $K_{2,t}$ minor is present in small graphs with the required number of edges.

Theorem 4.9 *Let t and d be positive integers, with $d < \max\left\{4, \frac{1}{11}t^{1/4}\right\}$. Let*

G be a graph of order $t+d$ with more than $\frac{t+1}{2}(|G|-1)$ edges. Then G has a $K_{2,t}$ minor.

Proof If $d<2$ the result is trivial, and if $2 \leq d \leq 3$ it is Lemma 4.5, so suppose $4 \leq d < \frac{1}{11}t^{1/4}$ and that the graph has no $K_{2,t}$ minor. Let G have maximum degree $t+s$. If $s > \frac{d-1}{2}$ or $s < \frac{d-3}{2}$, then let A be the set Y of Corollary 4.7 with X empty. Otherwise, let A be the set Y of Lemma 4.8 with X empty.

Now let $t+s'$ be the maximum degree in G of any vertex not in A. If $s' > \frac{d-1}{2}$ or $s' < \frac{d-3}{2}$, then let B be the set Y of Corollary 4.7 with $X=A$. Otherwise, let B be the set Y of Lemma 4.8 with $X=A$. Now A and B provide one half of the minor, and their common neighbours the other half. □

4.4 Large graphs

Lemma 4.10 *Let $t>200$ be a positive integer. Let G be a graph with average degree at least $t-3$. Suppose that G has a vertex v with degree at least $t+50(\log t)^2$, such that $G-v$ is connected. Suppose that there are at least $(t-3)/2$ triangles on every edge from v. Then G has a $K_{2,t}$ minor. Further, if $d(v) \geq \frac{5}{4}t$, then G has a $K_{2,1.03t}$ minor.*

Proof Let v have degree $\beta(t-3)$, where $\beta>1$. Every neighbour of v has at least $(t-3)/2$ neighbours in common with v. Thus, if u is any neighbour of v, and w is a random neighbour of v (chosen uniformly at random from $\Gamma(v)$), we have that $\mathbb{P}\big(u \notin \Gamma(w)\big) \leq 1-1/2\beta$. If (for some positive integer k) w_1, w_2, $\dots$, w_k are (not necessarily distinct) neighbours of v chosen uniformly and independently at random from $\Gamma(v)$, and we write $W = \{\, w_i : 1 \le i \le k \,\}$, then $\mathbb{P}\big(u \notin \Gamma(W)\big) \le (1 - 1/2\beta)^k < \exp(-k/2\beta)$.

If $\beta < 2$, let $k = \big\lceil 2\beta \log\big(\beta(t-3)\big)\big\rceil$. Then, for each $u \in \Gamma(v)$, we have $\mathbb{P}\big(u \notin \Gamma(W)\big) < |\Gamma(v)|^{-1}$. Thus, with positive probability, all vertices of $\Gamma(v)$ are neighbours of some vertex of W. Fix some such W. If $\beta \ge 2$, let $k = 3$. Then $\mathbb{P}\big(u \notin \Gamma(W)\big) \le (1 - 1/2\beta)^3 = 1 - 3/2\beta + 3/4\beta^2 - 1/8\beta^3 < 1 - 3/2\beta + 3/8\beta = 1 - 9/8\beta$. Thus, with positive probability, W has at least $\frac{9}{8}(t-3)$ neighbours in $\Gamma(v)$. Fix some such W.

$G - v$ is connected, so there are some paths in $G - v$ that connect W; clearly we may take such paths so that the path from w_i to w_j, if any, does not pass through any other element of W. Furthermore, if it contains more than one neighbour of either w_i or w_j, it may be shortened, and if it contains more than two neighbours of some other vertex $w_\ell \in W$, then it may be replaced by two paths, from w_i to w_ℓ and from w_ℓ to w_j, containing fewer interior vertices in total. Thus we arrive at a set of paths, such that the path from w_i to w_j contains at most one neighbour of each endpoint and at most two neighbours of each other element of W. There need only be $k - 1$ paths to form a spanning tree. Add the interior vertices of these paths to W to form W'. Then W has at most $(k-1)(2k-2)$ neighbours in $W' \setminus W$, so at most $(k-1)(2k-2) + k < 2k^2$ neighbours in W'.

If $\beta < 2$, we now observe that $k < 5 \log t$. Thus W has at least t neighbours in $\Gamma(v) \setminus W'$, yielding our $K_{2,t}$ minor. If $\beta < 2$ but $d(v) \ge \frac{5}{4}t \ge t + 50(\log t)^2$, then $t > 19000$ and $\frac{5}{4}t - 2k^2 > 1.03t$, yielding our $K_{2,1.03t}$ minor. If $\beta \ge 2$, observe that $(t-3)/8 - 2k^2 = t/8 - 18 - \frac{3}{8} > 0.03t$, so W has at least $1.03t$ neighbours in $\Gamma(v) \setminus W'$. □

If instead we had wished to find a $K_s + \overline{K_t}$ minor in the above lemma, we could have chosen $s - 1$ sets of vertices similarly to the set W above, and made them connected using the linking results of Bollobás and Thomason [4], provided that β is not too large.

Lemma 4.11 *Let* $t > 10^8$ *be a positive integer. Let* G *be a graph with average degree at least* $t+1$*, minimum degree at least* $(t+1)/2$*, at least* $(t-1)/2$ *triangles on every edge, and connectivity at least* $150 \log t$*. Let* $|G| \geq t + 300(\log t)^2$*. Then* G *has a* $K_{2,t}$ *minor.*

Proof Let v be a vertex of maximum degree. If $d(v) \geq t + 50(\log t)^2$, the result follows by Lemma 4.10, so we suppose $t+1 \leq d(v) < t + 50(\log t)^2$. If (with similar notation to the proof of Lemma 4.10) we put $d(v) = \beta(t-1)$, we have $\beta < 2$. Put $k = \lceil 2\beta \log\bigl(\beta(t-1)\bigr)\rceil$. As in that proof, choose W as k vertices taken independently at random from $\Gamma(v)$, and fix some particular W such that all vertices of $\Gamma(v)$ are neighbours of some vertex of W. (If this W happens to have fewer than k distinct vertices, add some arbitrary neighbours of v to W to make it up to k vertices.)

Now let y and z be any neighbours in $G - v - W$. Let $X = \Gamma(y) \cap \Gamma(z)$, so that $|X| \geq (t-1)/2$. Write $Y = \{v, y\}$ and $Z = W \cup \{z\}$. We wish to add some vertices to Y and Z such that each becomes connected. Enumerate Z as $z_1, z_2, \ldots, z_{k+1}$. The minimum degree of G is sufficient that for each i with $1 \leq i \leq k+1$ we may find $z_{i,1}$ and $z_{i,2}$ neighbours of z_i, all the $z_{i,j}$ being distinct and none of them being v or y. Now $k+1 < 5(\log t)^2$, so $G - Z$ is $22(k+1)$-connected, so $(k+1)$-linked in the sense of Definition 6.1, so we may find vertex-disjoint paths from v to y and from $z_{i,2}$ to $z_{i+1,1}$ for all i. This yields a path that may be added to Y to connect it, and paths that may be added to Z to connect that set.

As in the previous proof, we need to ensure that these paths consume few neighbours of the sets to which they are added. In the case of Y, the path may be shortened so that it contains at most one neighbour of each endpoint; letting the augmented set be Y', we then see that Y has at most $|Y|+2=4$ neighbours within Y'. In the case of Z, we end up with at most k paths, each containing at most $2k$ neighbours of vertices of Z, and adding these paths to make a set Z', so we have that Z has at most $|Z|+2k^2 = 2k^2+k+1$ neighbours within Z'.

Now, both Y' and Z' have as neighbours $\Gamma(v) \cup X$. If the graph does not have the required minor, it follows that $|\Gamma(v) \cup X| - 4 - (2k^2+k+1) < t$. Since $|\Gamma(v) \cup X| = |\Gamma(v)| + |X| - |\Gamma(v) \cap X| \geq \frac{3}{2}(t+1) - |\Gamma(v) \cap X|$, we must have $|\Gamma(v) \cap X| > (t-1)/2 - 2k^2 - k - 5 > d(v)/3$. But this means that every vertex of $G - v - \Gamma(v)$ has at least $d(v)/3$ neighbours in $\Gamma(v)$, so some vertex u of $\Gamma(v)$ has at least $|G - v - \Gamma(v)|/3$ neighbours in $G - v - \Gamma(v)$. But $|G - v - \Gamma(v)| \geq 250(\log t)^2 - 1$, so $|\Gamma(v) \cup \Gamma(u) - v| > t + 80(\log t)^2$. Contracting the edge between v and u leaves a graph satisfying the conditions of Lemma 4.10. □

To find a $K_s + \overline{K_t}$ minor above, s vertices could have been chosen in place of y and z.

Lemma 4.12 *Let $t > 10^{29}$ be a positive integer. Let G be a connected graph with more than $\frac{t+1}{2}(|G|-1)$ edges. Then G has a $K_{2,t}$ minor.*

Proof We work by induction on $|G|$. Note that $300(\log t)^2 < \frac{1}{11}t^{1/4}$. Thus, if $|G| < t + \frac{1}{11}t^{1/4}$, the result follows by Theorem 4.9, and otherwise we have $|G| \geq t + \frac{1}{11}t^{1/4} > t + 300(\log t)^2$.

If G has a vertex with degree less than or equal to $(t+1)/2$, remove it; if G has an edge on which there are fewer than $t/2$ triangles, contract it. These operations pass from G to a minor of G with fewer vertices, and do not decrease $e(G) - \frac{t+1}{2}|G|$. Thus we may suppose that G has minimum degree at least $(t+1)/2 + 1$ and at least $t/2$ triangles on every edge.

If $\kappa(G) \geq 150 \log t$, we are done by Lemma 4.11, so suppose $\kappa(G) < 150 \log t$. Let S be a cutset with $|S| = \kappa(G)$.

If $\kappa(G) = 1$, let X be some component of $G - S$. Both $G - X$ and $G[X \cup S]$ are minors of G with fewer vertices; if neither satisfies the conditions of the theorem, observe that together they have $e(G)$ edges, so that $e(G) \leq \frac{t+1}{2}(|G - X| - 1 + |X \cup S| - 1) = \frac{t+1}{2}(|G| - 1)$, a contradiction. Thus one of $G - X$ and $G[X \cup S]$ satisfies the conditions of the theorem.

It remains to consider the case of $2 \leq \kappa(G) < 150 \log t$. In this case, we may assume that $\Delta(G) < t + 50(\log t)^2$, since otherwise we may apply Lemma 4.10. If X is any component of $G - S$, and neither $G - X$ nor $G[X \cup S]$ satisfy the conditions of the theorem, we must have $e(G - X) \leq \frac{t+1}{2}(|G - X| - 1)$ and $e(G[X \cup S]) \leq \frac{t+1}{2}(|X \cup S| - 1)$. But then

$$\begin{aligned} e(G[X]) &\geq e(G) - e(G - X) - |S||X| \\ &\geq \tfrac{t+1}{2}(|G| - 1) + 1 - \tfrac{t+1}{2}(|G - X| - 1) - |S||X| \\ &= (\tfrac{t+1}{2} - |S|)|X| + 1 \\ &> (\tfrac{t+1}{2} - 150 \log t)|X|, \end{aligned}$$

for all components X of $G - S$. Each graph $G[X]$ must also have minimum degree at least $(t+1)/2 - 150 \log t$ and at least $t/2 - 150 \log t$ triangles on every edge.

Now let u and v be two vertices of S. We will find disjoint subsets U and V of X such that $G[U \cup \{u\}]$ and $G[V \cup \{v\}]$ are connected and $U \cup \{u\}$ and $V \cup \{v\}$ have at least $t/2$ common neighbours in $X - U - V$. Since $G - S$ has at least two components, we can then do the same with another component (with the same u and v) to find our minor.

Suppose x and $y \in X$ are neighbours. Then, by 2-connectivity, $G[X \cup \{u,v\}]$ has two vertex-disjoint paths from $\{u,v\}$ to $\{x,y\}$; without loss of generality, suppose that these paths are from u to x and from v to y. The path from u to x may be supposed to contain just one neighbour of x; that from v to y may be supposed to contain just one neighbour of y. Suppose we put the path from u to x in U, and that from v to y in V. Consider the common neighbours of x and y in X. At most one is in U, and at most one is in V. If they have as many as $t/2+2$ common neighbours in X, we have our minor, so suppose that $|\Gamma_X(x) \cap \Gamma_X(y)| \leq t/2+1$. This argument applies for any pair of neighbours in X, so we may suppose this inequality applies for all such pairs of neighbours.

If $|\Gamma_X(x)| + |\Gamma_X(y)| > 15t/8$, then $|\Gamma_X(x) \cup \Gamma_X(y) \setminus \{x,y\}| = |\Gamma_X(x)| + |\Gamma_X(y)| - |\Gamma_X(x) \cap \Gamma_X(y)| - 2 > 11t/8 - 3$. Contracting the edge xy, and contracting all components of $G-S$ other than X into S, we may then apply Lemma 4.10 to find a $K_{2,1.03(t-1000\log t)}$ minor. Since $1.03(t-1000\log t) > t$, we may now suppose that $|\Gamma_X(x)| + |\Gamma_X(y)| \leq 15t/8$ for all x, y neighbours in X.

$G[X]$ has average degree at least $t+1-300\log t$; that is, $2e(G[X]) = \sum_{x\in X} d_X(x) \geq (t+1-300\log t)|X|$. It follows that $\sum_{x\in X} d_X(x)^2 \geq (t+1-300\log t)\big(2e(G[X])\big)$. But $\sum_{x\in X} d_X(x)^2 = \frac{1}{2}\sum_{x\in X}\sum_{y\in\Gamma_X(x)}\big(d_X(x) + d_X(y)\big) \leq \frac{15}{16}\sum_{x\in X}\sum_{y\in\Gamma_X(x)} t = \frac{15}{16}t\big(2e(G[X])\big)$, a contradiction given the lower bound on t. □

To find a $K_s + \overline{K_t}$ minor above in an s-connected graph, paths would be taken from the cutset to more vertices than just x and y.

Of course, if G is not connected, we may just take some connected component of G with sufficiently many edges. We thus obtain the following result, which (considering the lower bound of Theorem 4.3) is best possible for $|G| \equiv 1 \pmod{t}$.

Theorem 4.13 *Let* $t > 10^{29}$ *be a positive integer. Let* G *be a graph with more than* $\frac{t+1}{2}(|G|-1)$ *edges. Then* G *has a* $K_{2,t}$ *minor.*

We now return to the more general problem of $K_{s,t}$ and $K_s + \overline{K_t}$ minors. Even for $s = 3$, we have no results better than the average degree $O\left(t\sqrt{\log t}\,\right)$ that forces a K_{s+t} minor, and so a $K_{s,t}$ minor. None of the methods of Section 4.3 apply to these more general minors. Many of the methods of Section 4.4 do apply more generally, but significant extra arguments would be needed to obtain useful results this way. For example, Lemma 4.10 can readily be extended if β is small, but when β is large there seems to be no simple way to apply it to $K_{s,t}$ minors for $s > 2$.

Chapter 5

Graphs without large complete minors

5.1 Introduction

Recall from Chapter 1 that Fernandez de la Vega [18] noticed from Bollobás, Catlin and Erdős [2] (see below) that random graphs are good examples of graphs with high average degree but no large complete minor. Kostochka [29, 30] showed that they are within a constant factor of being optimal. More recently, Thomason [64] essentially determined the extremal function for complete minors K_t in terms of the average degree, as $t \to \infty$: if we define

$$c(t) = \min\{\, c : e(G) \geq c|G| \text{ implies } K_t \prec G \,\}$$

then $c(t)$ exists and he showed that $c(t) = \big(\alpha + o(1)\big)t\sqrt{\log t}$, where $\alpha = 0.3190863431\ldots$ is an explicit constant; or, equivalently, that the minimum average degree guaranteeing a K_t minor is $\big(2\alpha + o(1)\big)t\sqrt{\log t}$.

Bollobás, Catlin and Erdős [2] showed that the largest K_t minor in a random graph $\mathcal{G}(n,p)$ has

$$t = \big(1+o(1)\big)\frac{n}{\sqrt{\log_{1/q} n}}$$

where $q = 1-p$. Choosing $q = \lambda = 0.2846681370\ldots$, another explicit constant, and $n = t\sqrt{\log_{1/\lambda} t}$, gives examples of graphs with average degree $\big(2\alpha + o(1)\big)t\sqrt{\log t}$ and no K_t minor. Examples with the same average degree and larger order are then constructed by taking many disjoint copies of $\mathcal{G}(n, 1-\lambda)$.

Thomason's proof in [64] therefore consists of showing that a graph (not necessarily random) of average degree greater than $\big(2\alpha + o(1)\big)t\sqrt{\log t}$ must have a K_t minor. Having proved this, he then claimed at the end of the paper, with an outline proof, that any extremal graph (that is, a graph with average degree $\big(2\alpha + o(1)\big)t\sqrt{\log t}$ and no K_t minor) is essentially the example given above: that (save for a few edges) it consists of a disjoint union of quasi-random graphs of the order and density given above. Here 'quasi-random' is used in the sense of Chung, Graham and Wilson [9] or Thomason [61]: that is, that every induced subgraph of order $|G|/2$ (or more generally $c|G|$ for any constant $0 < c < 1$) has essentially the same density.

Sós asked a more general question about complete minors and quasi-randomness. It is sometimes the case that quasi-random graphs contain larger minors than the corresponding random graphs; examples are given by Thomason [63], and indeed the problem, raised by Mader, of explicitly presenting graphs without large complete minors remains open. Sós asked whether, however, the converse might be true: that if a graph of order n and density p had no complete minor larger than that in a random graph $\mathcal{G}(n,p)$, would the graph then necessarily be quasi-random?

At first sight, the outline argument in Section 7 of [64] would appear to be usable to address Sós's question. The relevant part of the argument is, essentially, that if G is of maximal density having no K_t minor, then no subgraph of order $(1-\epsilon)|G|$ can have density much greater than that of G, or it would have a larger minor than that found in the whole of G. Thus G is quasi-random. (This uses a stronger result that a (reasonably connected) graph of density p has Hadwiger number at least that of a random graph of that density, to within a factor of $1+o(1)$.) This argument is, however, flawed on two counts: first, if the argument is quantified properly, using the method and results of [61], it turns out that the minor in the subgraph is not quite as large as is required; and second, the argument does not rule out the possibility of graphs G with very sparse subgraphs, and there are non-quasi-random graphs (such as some bipartite graphs) that have no large subgraph with significantly larger density than the original graph, but do have a few large subgraphs with significantly smaller density.

In this chapter, our purpose is to answer Sós's question; and at the same time, our results provide enough information to fill in the gaps in Thomason's argument.

The answer to Sós's question turns out to depend on the density and connectivity of G. A graph G of order n and density p that is not quasi-random will have a complete minor larger than that of a random graph $\mathcal{G}(n,p)$ if p is large (including $p \geq \frac{1}{2}$), and the same result holds for smaller p provided that G has moderate connectivity. Otherwise, if both the density and the connectivity are small, the assertion may fail; for example, the disjoint union of two $\mathcal{G}(n/2, \frac{1}{2})$ random graphs has order n and density $\frac{1}{4}$ but does not have a complete minor as large as that of $\mathcal{G}(n, \frac{1}{4})$.

The following notation will be useful in this chapter. Given a graph G whose vertex set is partitioned into two disjoint subsets X, Y, we define the three densities

$$p_X = \frac{e(X)}{\binom{|X|}{2}}, \qquad p_{XY} = \frac{e(X,Y)}{|X||Y|}, \qquad p_Y = \frac{e(Y)}{\binom{|Y|}{2}}$$

where $e(X)$, $e(Y)$ and $e(X,Y)$ are respectively the number of edges of G spanned by X, spanned by Y and joining X to Y. We likewise put $q_X = 1 - p_X$, $q_{XY} = 1 - p_{XY}$ and $q_Y = 1 - p_Y$. It is the principal feature of quasi-random graphs that, for every X of given order, the value of p_X differs little from p, the density of G itself, which of course implies that all of p_X, p_{XY} and p_Y are close to p.

A precise statement of the answer to Sós's question can now be given. This involves a constant

$$p_0 = \frac{1}{3}\left(4 + \sqrt[3]{3\sqrt{33} - 17} - \sqrt[3]{3\sqrt{33} + 17}\right) = 0.4563109873\ldots,$$

which is the real root of $x^3 - 4x^2 + 6x - 2 = 0$; and $q_0 = 1 - p_0$ is the real root of $x^3 + x^2 + x - 1 = 0$. (This arises from the inequality $q^4 - 2q + 1 = (q-1)(q^3 + q^2 + q - 1) > 0$; as long as this inequality holds, a random graph on half the vertices with twice the density will have a larger minor than a random graph on all the vertices, but when $q > q_0$ such a random graph on half the vertices will have a smaller minor, and the extremal graphs become the graphs made up of multiple disjoint random graphs with a few extra edges, described above, rather than being themselves random graphs.)

Theorem 5.1 *Given $\epsilon > 0$ there exist $\delta > 0$ and N such that the following assertion holds.*

Let G be a graph of order $n > N$ and edge density p, where $\epsilon < p < 1 - \epsilon$. Suppose that G has a vertex partition (X, Y) with $|X| = |Y|$ such that at least one of $|p_X - p|$, $|p_{XY} - p|$ and $|p_Y - p|$ exceeds ϵ. Suppose that either

$$p > p_0 + \epsilon \tag{5.1}$$

or

$$\kappa(G) \geq n(\log\log\log n)/(\log\log n). \tag{5.2}$$

Then G contains a K_t minor for

$$t > (1+\delta)\frac{n}{\sqrt{\log_{1/q} n}}$$

(where, as usual, $q = 1 - p$).

Roughly, this states that a non-quasi-random graph has a minor larger than a corresponding random graph provided that one of the conditions (5.1) or (5.2) holds.

In fact, provided we consider only graphs of reasonably connectivity (5.2), we can make a much more precise statement about the minimum order of a complete minor.

Let G be a graph of order n with a vertex partition (X, Y), where $|X| = \alpha|G|$. Let q_X, q_{XY}, q_Y be as above. Let $p = 1 - q$ be the density of G. Then, if n is large, we have essentially

$$q = \alpha^2 q_X + (1-\alpha)^2 q_Y + 2\alpha(1-\alpha) q_{XY}.$$

Consider now a constrained random graph G' of order n with a fixed vertex partition (X, Y), where the edges are chosen independently and at random, with probabilities p_X inside X, p_{XY} between X and Y and p_Y inside Y. It is straightforward to adapt the arguments of Bollobás, Catlin and Erdős [2] to show that the maximum order of a complete minor in this constrained random graph is

$$\big(1 + o(1)\big)\frac{n}{\sqrt{\log_{1/q_*} n}}$$

where

$$q_* = {q_X}^{\alpha^2} {q_Y}^{(1-\alpha)^2} {q_{XY}}^{2\alpha(1-\alpha)};$$

we saw in Section 2.3 that this is an upper bound on the Hadwiger number of such graphs. Taking logarithms and applying Jensen's inequality [26, 27], we see that

$$q \geq q_*,$$

with equality if and only if $q_X = q_Y = q_{XY}$.

The following theorem shows that our graph G with its given partition will have a complete minor at least as large as found in the corresponding constrained random graph G', provided that G has reasonable connectivity.

Theorem 5.2 *Let $0 < \epsilon < 1$. Then there exists N such that the following assertion holds.*

Let G be a graph of order $n > N$, with vertex partition (X, Y) as above, $|X| = \alpha n$, where $\epsilon < \alpha < 1 - \epsilon$. Let q_X, q_Y, q_{XY} and q_* be defined as above, and suppose $\epsilon < q_X, q_Y, q_{XY} \leq 1$ and $q_* < 1 - \epsilon$. Suppose $\kappa(G) \geq n(\log\log\log n)/(\log\log n)$. Then $G \succ K_s$, where

$$s = \left\lceil (1-\epsilon)\frac{n}{\sqrt{\log_{1/q_*} n}} \right\rceil.$$

This theorem is an extension of Theorem 4.1 of Thomason [64]; that theorem gives

$$s \geq (1-\epsilon)\frac{n}{\sqrt{\log_{1/q} n}},$$

when G has density p and reasonable connectivity; that theorem follows from Theorem 5.2 because $q \geq q_*$. The same inequality also means that Theorem 5.2 implies Theorem 5.1 for graphs of reasonable connectivity, except for extreme values of the parameters.

Much of this chapter is based on work published in *Combinatorics, Probability and Computing* as [39].

5.2 Outline of the proofs

We prove Theorem 5.2 first; then from it we derive Theorem 5.1. To prove Theorem 5.2, we must partition $V(G)$ into s parts $W_1, \ldots, W_s$, such that each $G[W_i]$ is connected and there is an edge in G between each W_i and W_j. The critical aspect is finding a partition that ensures that there are edges between each pair of parts of the minor; if such edges exist, the parts can be made connected, provided that G itself is reasonably connected.

For the case considered in Thomason [64], where all that is known about G is its density p (and that G is reasonably connected, where appropriate), that paper gives an argument for constructing a partition with the desired properties. The principal feature is to order the vertices of G by degree and to use this ordering to take a suitably constrained random partition.

At first sight it would appear that to extend this argument to the present case, where the existing partition (X, Y) and the densities p_X, p_Y and p_{XY} must be taken into account, would require a two-dimensional partial ordering of vertices by degrees to both X and Y; but such an argument is not strong enough to yield the required results. Nevertheless, somewhat surprisingly, it turns out that the argument can be adapted to the present case after all; although ordering the vertices by degree is not appropriate, there is a suitable function on the vertices which provides a single linear order that will work. Having found this ordering, the argument then follows somewhat similar lines to those of Thomason's proof of Theorem 4.1 in [64].

Having proved Theorem 5.2, Theorem 5.1 is derived as follows: either G is reasonably connected, in which case the result is immediate, or G has a very small cutset (and we require $q < q_0$ to go any further). If this cutset splits the graph into reasonably sized parts (each with at least $\frac{1}{50}$ of the vertices), we show that (for $q < q_0$) one of these parts is sufficiently much denser than the original graph that it would be expected to have a larger minor than a random graph of the same order and density as the original graph. If small cutsets only cut small numbers of vertices off the graph, we remove vertices of small degree; either only a few of them exist, so after removing them the resulting graph cannot have small parts cut off by small cutsets, or many exist, and after removing enough of them the resulting graph has a larger density. We iterate this process a bounded number of times, if necessary, ending up at a graph of large connectivity and with a large complete minor, and so deduce Theorem 5.1 using Theorem 5.2.

5.3 Proof of Theorem 5.2

We define a *complete equipartition* of G to be a partition of $V(G)$ into disjoint parts $W_1, \ldots, W_k$, such that G contains an edge from W_i to W_j for all $1 \leq i < j \leq k$ and such that $\lfloor |G|/k \rfloor \leq |W_i| \leq \lceil |G|/k \rceil$ for all i. The following lemma lies at the heart of this chapter.

Lemma 5.3 *Let G be a graph of order n with α, X, Y, q, q_X, q_Y, q_{XY}, q_* as above. Let ℓ, $s \geq 2$ be integers with $n = s\ell$ and $\ell\alpha$ an integer, $\alpha\ell \geq 2$, $(1-\alpha)\ell \geq 2$. Then G contains a complete equipartition into at least*

$$s - \frac{4s}{\omega\eta} - 2s^2(18\omega)^\ell \left[\frac{q_*}{1-\eta}\right]^{(1-\eta)\ell(\ell-\max\{1/\alpha,1/(1-\alpha)\})}$$

parts, for every $0 < \eta \leq 1 - {q_X}^{\alpha}{q_{XY}}^{(1-\alpha)}, 1 - {q_Y}^{(1-\alpha)}{q_{XY}}^{\alpha}$ and $\omega \geq 1$.

Proof For a vertex $v \in V(G)$ we define $Q(v;X) = \left\{ x \in X \setminus \{v\} : vx \notin E(G) \right\}$, the set of nonneighbours of v (other than v itself) within X, and $Q(v;Y) = \left\{ y \in Y \setminus \{v\} : vy \notin E(G) \right\}$, the set of nonneighbours of v (other than v itself) in Y. Also put $Q(v) = Q(v;X) \cup Q(v;Y)$. For $W \subset V(G)$, put

$N(W) = \{\, u \in V(G) : W \subset Q(u) \,\}$. Let

$$q(v;X) = |Q(v;X)|/(\alpha n - 1),$$

$$q(v;Y) = |Q(v;Y)|/\big((1-\alpha)n - 1\big).$$

Put

$$r(v) = q(v;X)^{\alpha\ell} q(v;Y)^{(1-\alpha)\ell}.$$

Then order the vertices of X as $x_1, \dots, x_{\alpha n}$ in order of increasing $r(x_i)$, and similarly order the vertices of Y as $y_1, \dots, y_{(1-\alpha)n}$ in order of increasing $r(y_i)$.

Now define blocks $B_j^X = \{\, x_i : (j-1)s < i \le js \,\}$ for $1 \le j \le \alpha\ell$, and $B_j^Y = \{\, y_i : (j-1)s < i \le js \,\}$ for $1 \le j \le (1-\alpha)\ell$. Independently and uniformly choose random permutations β_j^X, β_j^Y of the blocks, and so induce a random partition of $V(G)$ into s parts $W_t = \{\, x_{\beta_j^X(t)} : 1 \le j \le \alpha\ell \,\} \cup \{\, y_{\beta_j^Y(t)} : 1 \le j \le (1-\alpha)\ell \,\}$, $1 \le t \le s$.

Let $S^X \subset X$, $S^Y \subset Y$, $S = S^X \cup S^Y$. Then, for W one of the random parts,

$$\begin{aligned}
\mathbb{P}(W \subset S) &= \prod_{j=1}^{\alpha\ell} \frac{|S^X \cap B_j^X|}{s} \prod_{j=1}^{(1-\alpha)\ell} \frac{|S^Y \cap B_j^Y|}{s} \\
&\le \left[\frac{1}{\alpha\ell} \sum_{j=1}^{\alpha\ell} \frac{|S^X \cap B_j^X|}{s}\right]^{\alpha\ell} \left[\frac{1}{(1-\alpha)\ell} \sum_{j=1}^{(1-\alpha)\ell} \frac{|S^Y \cap B_j^Y|}{s}\right]^{(1-\alpha)\ell} \\
&= \left[\frac{|S^X|}{\alpha n}\right]^{\alpha\ell} \left[\frac{|S^Y|}{(1-\alpha)n}\right]^{(1-\alpha)\ell},
\end{aligned}$$

using the AM/GM inequality.

For $S = Q(x_i)$, we have

$$\mathbb{P}\big(x_i \in N(W)\big) = \mathbb{P}(W \subset S) \le q(x_i;X)^{\alpha\ell} q(x_i;Y)^{(1-\alpha)\ell} = r(x_i).$$

Similarly, $\mathbb{P}\big(y_i \in N(W)\big) \le r(y_i)$. By the ordering of vertices chosen,

$$\mathbb{E}\left(|B_j^X \cap N(W)|\right) \le s r(x_{js}),$$

and

$$\mathbb{E}\left(|B_j^Y \cap N(W)|\right) \le sr(y_{js}).$$

Say that W *rejects* a block B_j^X (respectively B_j^Y) if $|B_j^X \cap N(W)| > \omega sr(x_{js})$ (respectively $|B_j^Y \cap N(W)| > \omega sr(y_{js})$), so that W rejects a given block with probability at most $1/\omega$; put $R^X(W) = \{ j < \alpha\ell : W \text{ rejects } B_j^X \}$ and $R^Y(W) = \{ j < (1-\alpha)\ell : W \text{ rejects } B_j^Y \}$, so $\mathbb{E}\left(|R^X(W)|\right) \le (\alpha\ell - 1)/\omega$ and $\mathbb{E}\left(|R^Y(W)|\right) \le \big((1-\alpha)\ell - 1\big)/\omega$. Call a random part W *acceptable* if $|R^X(W)| < \eta(\alpha\ell - 1)$ and $|R^Y(W)| < \eta\big((1-\alpha)\ell - 1\big)$, so

$$\mathbb{P}(W \text{ is not acceptable}) < 2/\omega\eta.$$

Now let W be some acceptable part; put $M^X(W) = \{1, \dots, \alpha\ell - 1\} \setminus R^X(W)$, $M^Y(W) = \{1, \dots, (1-\alpha)\ell - 1\} \setminus R^Y(W)$, $m^X = |M^X(W)| \ge (1-\eta)(\alpha\ell - 1)$ and $m^Y = |M^Y(W)| \ge (1-\eta)\big((1-\alpha)\ell - 1\big)$. Let W' be another random part and let P_W be the probability, conditional on W, of there being no edge from W' to W. Then we have

$$\begin{aligned} P_W &= \mathbb{P}\big(W' \subset N(W) \bigm| W\big) \\ &\le \prod_{j \in M^X(W)} \frac{\omega sr(x_{js})}{s-1} \prod_{j \in M^Y(W)} \frac{\omega sr(y_{js})}{s-1} \\ &< (2\omega)^\ell \prod_{j \in M^X(W)} r(x_{js}) \prod_{j \in M^Y(W)} r(y_{js}). \end{aligned}$$

Now, we have

$$\begin{aligned} \left[\prod_{j \in M^X(W)} r(x_{js})^{1/\ell}\right]^{1/m^X} &\le \frac{1}{m^X} \sum_{j \in M^X(W)} r(x_{js})^{1/\ell} \\ &= \frac{1}{m^X} \sum_{j \in M^X(W)} q(x_{js}; X)^\alpha q(x_{js}; Y)^{(1-\alpha)} \\ &\le \frac{1}{m^X s} \sum_{i=1}^{\alpha n} q(x_i; X)^\alpha q(x_i; Y)^{(1-\alpha)} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{m^X s}\left[\sum_{i=1}^{\alpha n} q(x_i;X)\right]^{\alpha}\left[\sum_{i=1}^{\alpha n} q(x_i;Y)\right]^{(1-\alpha)} \\
&= \frac{\alpha n {q_X}^{\alpha} {q_{XY}}^{(1-\alpha)}}{m^X s} \\
&\leq \frac{{q_X}^{\alpha} {q_{XY}}^{(1-\alpha)}}{1-\eta} \times \frac{\alpha\ell}{\alpha\ell - 1}
\end{aligned}
$$

(using Hölder's inequality [24]) and likewise

$$
\left[\prod_{j\in M^Y(W)} r(y_{js})^{1/\ell}\right]^{1/m^Y} \leq \frac{{q_Y}^{(1-\alpha)} {q_{XY}}^{\alpha}}{1-\eta} \times \frac{(1-\alpha)\ell}{(1-\alpha)\ell - 1}
$$

whence

$$
\begin{aligned}
P_W &\leq (2\omega)^{\ell}\left[\frac{{q_X}^{\alpha} {q_{XY}}^{(1-\alpha)}}{1-\eta} \times \frac{\alpha\ell}{\alpha\ell - 1}\right]^{\ell m^X} \\
&\quad \times \left[\frac{{q_Y}^{(1-\alpha)} {q_{XY}}^{\alpha}}{1-\eta} \times \frac{(1-\alpha)\ell}{(1-\alpha)\ell - 1}\right]^{\ell m^Y} \\
&\leq (18\omega)^{\ell}\left[\frac{q_*}{1-\eta}\right]^{(1-\eta)\ell(\ell-\max\{1/\alpha,1/(1-\alpha)\})} \\
&= P,
\end{aligned}
$$

say.

Now, we have a partition with at most $4s/\omega\eta$ unacceptable parts and at most $2s^2P$ defective pairs of acceptable parts with no edge between them. Remove each unacceptable part, and one part from each defective pair. This yields an equipartition of part of the graph into the required number of parts, and the remaining vertices may then be distributed among those parts. □

We now convert this lemma into a more usable form.

Lemma 5.4 *Let* $0 < \epsilon < 1$. *Then there exists* N *such that the following assertion holds.*

Let G *be a graph of order* $n > N$, *with vertex partition* (X, Y), $|X| = \beta n$, *where* $\epsilon < \beta < 1 - \epsilon$. *Let* $\epsilon < q_X, q_Y, q_{XY}$ *and* $q_* < 1 - \epsilon$. *Then* G *has a complete equipartition into at least* $(1-\epsilon)n/\sqrt{\log_{1/q_*} n}$ *parts.*

Proof Suppose n large (sufficiently large for all the parts of this proof to work). Put $d = \lfloor\sqrt{n}\rfloor$. We apply Lemma 5.3 with $\alpha = \lfloor d\beta\rfloor/d$, $\ell = d\left\lceil(1/d)(1+\epsilon/2)\sqrt{\log_{1/q_*} n}\,\right\rceil$, $s = \lfloor n/\ell\rfloor$, $\eta = \epsilon(1-q_*)/8$ and $\omega = 128/\epsilon^2(1-q_*)$. We lose a few vertices from G in the conversion to integer s and ℓ, but only $O\left(\sqrt{\log_{1/q_*} n}\right) < \epsilon^3 n$ of them, so the effect on the n and q_* used in Lemma 5.3 is insignificant.

We have $s > (1-\epsilon/2)n/\sqrt{\log_{1/q_*} n}$, so it will suffice to show that each of the terms subtracted from s in the statement of Lemma 5.3 is at most $\epsilon s/4$; this holds for the first term by choice of η and ω. For the second, we have $\ell\big(\ell - \max\{1/\alpha, 1/(1-\alpha)\}\big) > (1+\epsilon)\log_{1/q_*} n$, and since $\eta < \epsilon/8$ we have $(1-\eta)\ell\big(\ell-\max\{1/\alpha, 1/(1-\alpha)\}\big) > (1+3\epsilon/4)\log_{1/q_*} n$. Also, $\log\big(1/(1-\eta)\big) = -\log(1-\eta) < 2\eta = \epsilon(1-q_*)/4$ since $\eta < \frac{1}{8}$; and $1 - q_* < \log(1/q_*)$, so $\log\big(1/(1-\eta)\big) < (\epsilon/4)\log(1/q_*)$; thus $\log\big(q_*/(1-\eta)\big) < (\epsilon/4 - 1)\log(1/q_*)$. Thus,

$$
\begin{aligned}
2s^2(18\omega)^\ell &\left[\frac{q_*}{1-\eta}\right]^{(1-\eta)\ell(\ell-\max\{1/\alpha,1/(1-\alpha)\})} \\
&\leq s\exp\left[\log n + \ell\log\left(2304/\epsilon^2(1-q_*)\right) - (1+3\epsilon/4)(1-\epsilon/4)\log n\right] \\
&\leq s\exp\left[2\sqrt{\log_{1/q_*} n}\,\log\left(2304/\epsilon^2(1-q_*)\right) - (\epsilon/4)\log n\right] \\
&\leq s\exp\left[2\sqrt{(\log n)/(1-q_*)}\,\log\left(2304/\epsilon^2(1-q_*)\right) - (\epsilon/4)\log n\right] \\
&< \epsilon s/4
\end{aligned}
$$

for n large, given the bounds on q_*. □

We now use this result to find complete minors in dense graphs. We use the results of Section 3.2.

Proof of Theorem 5.2 Assume throughout that n is large. By Lemma 3.4, for any u, $v \in V(G)$, u and v are joined in G by at least $\kappa^2/4n$ internally disjoint paths with length at most

$$h = 2(\log\log n)/(\log\log\log n);$$

let $P_{u,v}$ be the set of such paths.

Let $r = 1/(\log\log\log n)$ and select vertices independently and at random with probability r from $V(G)$, forming a set of vertices C, where $|C| < 2rn$ with probability at least $\frac{1}{2}$. Using Lemma 3.3, the probability that a given vertex $v \in G$ of degree $d(v)$ has more than $\epsilon d(v)/6$ neighbours within C is less than $1/n^2$. For given u, $v \in V(G)$, C contains all the internal vertices of some given path in $P_{u,v}$ with probability at least r^h, independently for each such path; and $r^h > (\log n)^{-1/6}$, so $r^h|P_{u,v}|/2 > n/(\log n)^{1/3}$. Again using Lemma 3.3, we conclude that the probability that fewer than $r^h|P_{u,v}|/2$ paths of $P_{u,v}$ lie entirely within C is less than $1/n^3$; so there is some set C (which we now fix) with $|C| < 2rn$, with every vertex v of G having at most $\epsilon d(v)/6$ neighbours inside C, and, for every pair u, v of vertices of G, with at least $n/(\log n)^{1/3}$ internally disjoint paths from u to v with length at most h whose internal vertices lie within C.

Similarly, choose a random subset D of $V(G) \setminus C$, choosing each vertex with probability r. With probability at least $\frac{1}{2}$ we have $|D| < 2rn$; any given vertex v has at least $d(v)/2 \geq \kappa/2$ neighbours outside C and the probability that more than $\epsilon d(v)/6$ of these or fewer than $r\kappa/4$ of these lie in D is at most $1/n^2$; so we may fix D such that every vertex v has between $r\kappa/4$ and $\epsilon d(v)/6$ neighbours in D.

Now consider the graph $G - C - D$, and apply Lemma 5.4 to it with parameter $\epsilon/8$. Each of q_X, q_Y, q_{XY} has changed by at most $\epsilon^2/10$, so we may find a complete equipartition of $G - C - D$ into s parts, say $W'_1, \ldots, W'_s$. Now G, K_s, C and D satisfy the conditions of Lemma 3.2, so by that result we have our minor. □

5.4 Proof of Theorem 5.1

From now on, we aim only for minors of order $(1+\delta)n/\sqrt{\log_{1/q} n}$, not for stronger results involving q_*. Theorem 5.2 now yields Theorem 5.1 in the well-connected case.

Lemma 5.5 *Let $\epsilon > 0$ be given. Then there exist $\delta > 0$ and N such that the following assertion holds.*

Let G be a graph of order $n > N$ and edge density p, where $\epsilon < p < 1-\epsilon$. Suppose that G has a vertex partition (X, Y) with $|X| = |Y|$, such that at least one of $|p_X - p|$, $|p_{XY} - p|$ and $|p_Y - p|$ exceeds ϵ. Suppose that $\kappa(G) \geq n(\log\log\log n)/(\log\log n)$. Then G contains a K_t minor for $t > (1+\delta)n/\sqrt{\log_{1/q} n}$ (where, as usual, $q = 1-p$).

Proof Since $\log q = \log\bigl(\alpha^2 q_X + 2\alpha(1-\alpha)q_{XY} + (1-\alpha)^2 q_Y\bigr)$ and $\log q_* = \alpha^2 \log q_X + 2\alpha(1-\alpha)\log q_{XY} + (1-\alpha)^2 \log q_Y$, we can, by considering the graph of $\log x$, choose small ϵ_1 (much smaller than ϵ) and $\delta > 0$ such that if $q \geq \epsilon/2$ and if any of $|q_X - q|$, $|q_Y - q|$, $|q_{XY} - q|$ exceeds $\epsilon/4$, then $(1-\epsilon_1)\sqrt{\log(1/q_{**})} > (1+\delta)\sqrt{\log(1/q)}$ holds, where we define $q_{**} = \max\{\epsilon_1, q_X\}^{\alpha^2} \max\{\epsilon_1, q_Y\}^{(1-\alpha)^2} \max\{\epsilon_1, q_{XY}\}^{2\alpha(1-\alpha)}$.

If, now, $\epsilon_1 < q_X$, q_Y, q_{XY}, this lemma follows by applying Theorem 5.2 to G with ϵ_1 in place of ϵ. If we have one of $q_X, q_Y, q_{XY} \leq \epsilon_1$ (but nevertheless $q > \epsilon$), then this means that almost all edges are present in some part of the graph, and q_* is much smaller than q. Remove a few edges from the relevant part or parts of the graph to increase q_X, q_Y, q_{XY} to above ϵ_1; by a result of Mader [37] that a minimal k-connected graph on n vertices ($n \geq 3k$) has at most $k(n-k)$ edges, we may easily do this while preserving the required connectivity. Since ϵ_1 is small compared to q, after removing these edges, we still have (in the modified graph) one of $|q_X - q|$, $|q_Y - q|$, $|q_{XY} - q|$ exceeding $\epsilon/4$, so Theorem 5.2 applied to the new graph gives our result. □

It now remains only to consider the case of small connectivity. Define the expected order of a complete minor in a random graph of order n and density of nonedges q to be $t(n,q) = n/\sqrt{\log_{1/q} n}$. In many cases, we will reduce from a graph G of order n and density at least $p = 1-q$ to a subgraph H of order βn, and want the expected order of a complete minor in H to be as large as that expected in a random graph of order n and edge density at least p; that is, if H is of density $p' = 1-q'$, we will want $\beta n \Big/ \sqrt{\log_{1/q'}(\beta n)} \geq n/\sqrt{\log_{1/q} n}$; it will suffice if $\beta\sqrt{\log(1/q')} \geq \sqrt{\log(1/q)}$, that is, if $q' \leq q^{1/\beta^2}$. Define $q'(q,\beta) = q^{1/\beta^2}$. Similarly, we may want H to have a minor at least $(1+\delta)$ times larger, so we also define $q'(q,\beta,\delta) = q^{(1+\delta)^2/\beta^2}$.

Lemma 5.6 *Let* $f_q(\alpha) = 1-\alpha^2-(1-\alpha)^2+\alpha^2 q^{1/\alpha^2}+(1-\alpha)^2 q^{1/(1-\alpha)^2}-q$. *If* $0 < \alpha < 1$ *and* $0 \leq q < q_0 = 1-p_0$, *then* $f_q(\alpha) > 0$. *Further, for* $0 \leq q < q_0$, *we have* $f_q(\frac{1}{100}) > 10^{-3}$.

Proof The behaviour of the function $f_q(\alpha)$ is illustrated by Figure 5.1, in which graphs of $f_{0.4}$, $f_{0.5}$ and $f_{0.55}$ are shown. A quick glance at this figure makes the lemma appear very plausible. Unfortunately, I don't have a short and elegant proof of the lemma. The proof here involves computer verification of many cases; the tables of cases, and the source code for the program that generated them and so completes the verification of the result, are in Appendix A.

Observe first that for $\alpha = \frac{1}{2}$, $f_q(\alpha) = \frac{1}{2} + \frac{1}{2}q^4 - q$. Thus $f_q(\frac{1}{2}) > 0$ if and only if $q^4 - 2q + 1 = (q-1)(q^3+q^2+q-1) > 0$, which on $0 < q < 1$ occurs just when $q > q_0$. Note also that $f_q(0) = f_q(1) = 0$. Clearly, f_q is symmetrical about $\alpha = \frac{1}{2}$.

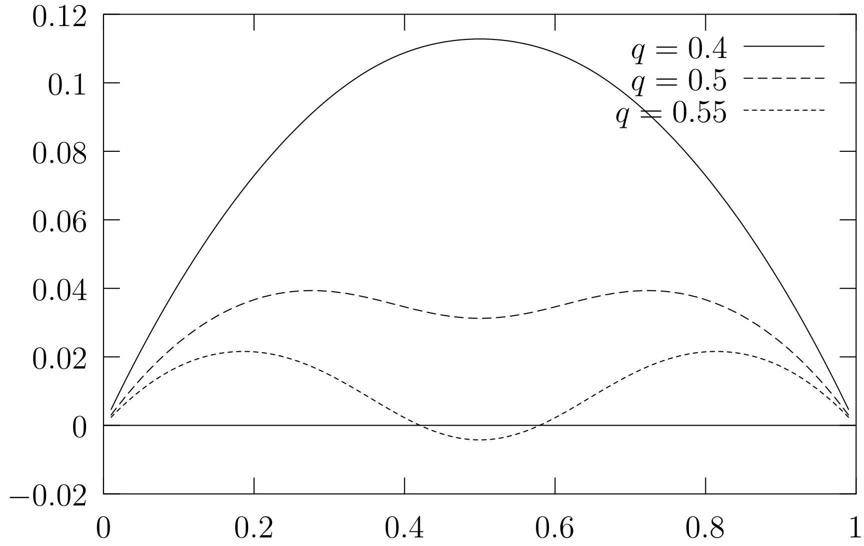

Figure 5.1: f_q

We have

$$\frac{\partial f}{\partial \alpha} = -4\alpha+2+q^{1/\alpha^2}(2\alpha-2\alpha^{-1}\log q)-q^{1/(1-\alpha)^2}\left(2(1-\alpha)-2(1-\alpha)^{-1}\log q\right)$$

and so

$$\begin{aligned}\frac{\partial^2 f}{\partial \alpha^2} &= -4+q^{1/\alpha^2}\left(2-2\alpha^{-2}\log q+4\alpha^{-4}(\log q)^2\right)\\ &\quad +q^{1/(1-\alpha)^2}\left(2-2(1-\alpha)^{-2}\log q+4(1-\alpha)^{-4}(\log q)^2\right)\end{aligned}$$

and all these functions are well-defined and continuous in α for any $0 \le q < 1$.

The lemma may now be proved by numeric computation. The function $-r\log r$ on $[0,1]$ is zero at 0 and 1, and has a unique maximum at $r = e^{-1}$. The function $r(\log r)^2$ on $[0,1]$ is zero at 0 and 1, and has a unique maximum at $r = e^{-2}$. The second derivative above is composed of constants, q^{1/α^2}, these functions for $r = q^{1/\alpha^2}$, and corresponding functions of $1-\alpha$. Thus, given bounds on q and α, we may deduce bounds on the second partial derivative. To be precise, given $r_{\min} \le r \le r_{\max}$, we have

$$\min\{-r_{\min}\log r_{\min}, -r_{\max}\log r_{\max}\} \le -r\log r \le e^{-1}$$

if $r_{\min} \le e^{-1} \le r_{\max}$, and

$$\begin{aligned} \min\{-r_{\min}\log r_{\min}, -r_{\max}\log r_{\max}\} &\le -r\log r \\ &\le \max\{-r_{\min}\log r_{\min}, -r_{\max}\log r_{\max}\} \end{aligned}$$

otherwise; similarly, we have

$$\min\{r_{\min}(\log r_{\min})^2, r_{\max}(\log r_{\max})^2\} \le r(\log r)^2 \le 4e^{-2}$$

if $r_{\min} \le e^{-2} \le r_{\max}$, and

$$\begin{aligned} &\min\{r_{\min}(\log r_{\min})^2, r_{\max}(\log r_{\max})^2\} \\ &\qquad \le r(\log r)^2 \\ &\qquad \le \max\{r_{\min}(\log r_{\min})^2, r_{\max}(\log r_{\max})^2\} \end{aligned}$$

otherwise. If we now have that $q_{\min} \le q \le q_{\max}$ and $\alpha_{\min} \le \alpha \le \alpha_{\max}$, we consider these bounds applied to q^{1/α^2} and $q^{1/(1-\alpha)^2}$, which satisfy

$$q_{\min}^{1/\alpha_{\min}^2} \le q^{1/\alpha^2} \le q_{\max}^{1/\alpha_{\max}^2}$$

and

$$q_{\min}^{1/(1-\alpha_{\max})^2} \le q^{1/(1-\alpha)^2} \le q_{\max}^{1/(1-\alpha_{\min})^2}.$$

At $\alpha = \frac{1}{2}$, the first partial derivative is zero. For $0 \le q \le 0.4$, we prove the lemma by showing that the second partial derivative is negative for $0 \le \alpha \le \frac{1}{2}$. This may be done by considering the numerical bounds for four regions of (q, α) space, being $0 \le \alpha \le \frac{1}{4}$; $\frac{1}{4} \le \alpha \le \frac{3}{8}$; $\frac{3}{8} \le \alpha \le \frac{7}{16}$; and $\frac{7}{16} \le \alpha \le \frac{1}{2}$, each being for the whole range $0 < q \le 0.4$. The bounds are shown in Table A.1.

For $0.4 \le q \le 0.48$, determine the bounds on the second partial derivative for regions of $\frac{1}{32}$ in α. These then give bounds on the first partial derivative (working from $\frac{1}{2}$ to 0), which then bound $f_q(\alpha)$ itself, showing it to be greater than 0 for $0 < \alpha < 1$. Given bounds $\partial^2_{\min}[(r-1)/32, r/32]$ and $\partial^2_{\max}[(r-1)/32, r/32]$ on the second partial derivative for $(r-1)/32 \le \alpha \le r/32$, and bounds $\partial_{\min}(r/32)$ and $\partial_{\max}(r/32)$ on the first partial derivative for $\alpha = r/32$, we compute $\partial_{\min}\big((r-1)/32\big) = \partial_{\min}(r/32) - \partial^2_{\max}[(r-1)/32, r/32]/32$ and $\partial_{\max}\big((r-1)/32\big) = \partial_{\max}(r/32) - \partial^2_{\min}[(r-1)/32, r/32]/32$; and given these formulas, bounds on the first partial derivative for $(r-1)/32 \le \alpha \le r/32$ may be found as $\partial_{\min}[(r-1)/32, r/32] = \min\big\{\partial_{\min}\big((r-1)/32\big), \partial_{\min}(r/32)\big\}$ and $\partial_{\max}[(r-1)/32, r/32] = \max\big\{\partial_{\max}\big((r-1)/32\big), \partial_{\max}(r/32)\big\}$. Similarly, we then compute bounds on $f_q(\alpha)$, starting from $f_q(0) = 0$: we find successively that $f_{\min}\big((r+1)/32\big) = f_{\min}(r/32) + \partial_{\min}[r/32, (r+1)/32]/32$, $f_{\max}((r+1)/32) = f_{\max}(r/32) + \partial_{\max}[r/32, (r+1)/32]/32$; $f_{\min}[r/32, (r+1)/32] = \min\big\{f_{\min}(r/32), f_{\min}\big((r+1)/32\big)\big\}$ and $f_{\max}[r/32, (r+1)/32] = \max\big\{f_{\max}(r/32), f_{\max}\big((r+1)/32\big)\big\}$. The main bounds (from which the others may trivially be derived) are shown in Table A.2.

Finally, on the regions $0.48 \le q \le 0.5$ and $0.5 \le q \le 0.55$ separately, we show that f has the expected shape so that it cannot be zero anywhere if $f_q(\frac{1}{2}) > 0$. That is, we show (using steps of $\frac{1}{64}$ in α) that, for some $0 < \alpha_0 < \frac{1}{2}$, we have the second partial derivative negative for $\alpha \le \alpha_0$, and that the first partial derivative is negative for $\alpha_0 \le \alpha < \frac{1}{2}$. The first partial derivative is positive at $\alpha = 0$, and so we have a single minimum at $\alpha = \frac{1}{2}$. The main bounds are shown in Table A.3 (for $0.48 \le q \le 0.5$) and Table A.4 (for $0.5 \le q \le 0.55$).

It remains to show the lower bound on $f_q(\frac{1}{100})$. For this, observe

$$\frac{\partial f}{\partial \alpha} = -4\alpha + 2 + q^{1/\alpha^2}(2\alpha - 2\alpha^{-1}\log q)$$

$$\begin{aligned}
&\quad - q^{1/(1-\alpha)^2}\big(2(1-\alpha) - 2(1-\alpha)^{-1}\log q\big) \\
&\geq -4\alpha + 2 - q^{1/(1-\alpha)^2}\big(2(1-\alpha) - 2(1-\alpha)^{-1}\log q\big) \\
&\geq -4\alpha + 2 - q\big(2(1-\alpha) - 2(1-\alpha)^{-1}\log q\big) \\
&\geq -4\alpha + 2 - 2q + 2(1-\alpha)^{-1} q \log q \\
&\geq -4\alpha + 2 - 2q_0 - 2(1-\alpha)^{-1}e^{-1}.
\end{aligned}$$

This last expression is decreasing in α, and for $\alpha = \frac{1}{100}$ this value is positive, so $f_q(\frac{1}{100}) \geq \frac{1}{100}\big(-4 \times \frac{1}{100} + 2 - 2q_0 - 2(1-\frac{1}{100})^{-1}e^{-1}\big) > 10^{-3}$. □

We now apply this lemma.

Corollary 5.7 *Let $\epsilon > 0$ be given. Then there exist $\delta > 0$ and N such that the following assertion holds.*

Let G be a graph of order $n > N$ and edge density at least p, where $p_0 + \epsilon < p$. Suppose $\kappa(G) < n(\log\log\log n)/(\log\log n)$, and that there exists a cutset S in G with $|S| = \kappa(G)$ such that there exist X, Y with $V(G) = X \cup Y$, $S = X \cap Y$ and $E(G) = E(G[X]) \cup E(G[Y])$, and $\frac{1}{100}(n + |S|) \leq |X| \leq \frac{99}{100}(n + |S|)$. Then G has a subgraph H of order at least $\frac{1}{100}n$ and at most $\frac{99}{100}(n + |S|)$ and density $p' = 1 - q'$ where $q' \leq q'(q, |H|/n, \delta)$.

Proof Suppose we have such a cutset, and let $|S| = \gamma n$. Choose our X, Y. Our subgraph H will be one of $G[X]$ and $G[Y]$. Put $|X| = \alpha(1+\gamma)n$ and $|Y| = (1-\alpha)(1+\gamma)n$, where $\frac{1}{100} \leq \alpha \leq \frac{99}{100}$.

Define p_X, p_Y accordingly as the densities of edges in X, Y; so that $p \leq \alpha^2(1+\gamma)^2 p_X + (1-\alpha)^2(1+\gamma)^2 p_Y$ and $q \geq 1 - \alpha^2(1+\gamma)^2(1-q_X) - (1-\alpha)^2(1+\gamma)^2(1-q_Y) = \big(1 - \alpha^2(1+\gamma)^2 - (1-\alpha)^2(1+\gamma)^2\big) + \alpha^2(1+\gamma)^2 q_X + (1-\alpha)^2(1+\gamma)^2 q_Y = s$, say.

We want to show that either $q_X \leq q'\big(q, \alpha(1+\gamma), \delta\big)$ or $q_Y \leq q'\big(q, (1-\alpha)(1+\gamma), \delta\big)$. Since we have $q \geq s$, it will suffice to show that either $q_X \leq$

$q'\big(s,\alpha(1+\gamma),\delta\big)$ or $q_Y \le q'\big(s,(1-\alpha)(1+\gamma),\delta\big)$. Suppose not; we shall derive a contradiction. For, we then have $q_X > q'\big(s,\alpha(1+\gamma),\delta\big)$ and $q_Y > q'\big(s,(1-\alpha)(1+\gamma),\delta\big)$, so $s > \big(1-\alpha^2(1+\gamma)^2-(1-\alpha)^2(1+\gamma)^2\big)+\alpha^2(1+\gamma)^2 q'\big(s,\alpha(1+\gamma),\delta\big) + (1-\alpha)^2(1+\gamma)^2 q'\big(s,(1-\alpha)(1+\gamma),\delta\big)$, that is, $f(s,\alpha,\gamma,\delta) = \big(1-\alpha^2(1+\gamma)^2-(1-\alpha)^2(1+\gamma)^2\big)+\alpha^2(1+\gamma)^2 s^{(1+\delta)^2/\alpha^2(1+\gamma)^2} + (1-\alpha)^2(1+\gamma)^2 s^{(1+\delta)^2/(1-\alpha)^2(1+\gamma)^2} - s \le 0$. This function is continuous in all four variables, and $f(s,\alpha,0,0)$ is $f_s(\alpha)$ in the notation of Lemma 5.6.

By Lemma 5.6, $f_s(\alpha)$ is bounded away from zero on $\frac{1}{100} \le \alpha \le \frac{99}{100}$, $0 \le q \le q_0 - \epsilon$. By continuity (and so uniform continuity), we deduce that we cannot have $f(s,\alpha,\gamma,\delta) \le 0$ for γ, δ sufficiently small (depending on ϵ), so providing our contradiction. □

Corollary 5.8 *Let $\epsilon > 0$ be given. Then there exist $\delta > 0$ and N such that the following assertion holds.*

Let G be a graph of order $n > N$ and edge density at least p, where $p_0 + \epsilon < p < 1 - \epsilon$. Suppose that G has a vertex partition (X', Y') with $|X'| = |Y'|$, such that at least one of $|p_{X'} - p|$, $|p_{X'Y'} - p|$ and $|p_{Y'} - p|$ exceeds ϵ. Suppose that $\delta(G) \ge \frac{1}{60}n$. Then either G contains a K_t minor for $t > (1+\delta)n/\sqrt{\log_{1/q} n}$ (where, as usual, $q = 1-p$) or G has a subgraph H of order at least $\frac{1}{100}n$ and at most $\frac{199}{200}n$ and density $p' = 1 - q'$ where $q' \le q'(q, |H|/n, \delta)$.

Proof If $\kappa(G) \ge n(\log\log\log n)/(\log\log n)$, we have a large minor by Lemma 5.5. Otherwise, we have a small cutset S, with $|S| = \kappa(G)$, and if we choose any division of G by this cutset, this induces X, Y satisfying the conditions of Corollary 5.7 (since, for any choice of X, Y, where one of X and Y might be too small, some vertex in X has degree at most $|X|$; but the bound on the minimal degrees then implies that $|X|$, $|Y| \geq \frac{1}{60}n$). The result then follows by Corollary 5.7. $\square$

Corollary 5.9 *Let $\epsilon > 0$ be given. Then there exists N such that the following assertion holds.*

Let G be a graph of order $n > N$ and edge density at least p, where $p_0 + \epsilon < p < 1 - \epsilon$. Suppose $\delta(G) \geq \frac{1}{50}n$. Then either G contains a K_t minor for $t > (1-\epsilon)n/\sqrt{\log_{1/q} n}$ (where, as usual, $q = 1 - p$) or G has a subgraph H of order at least $\frac{1}{100}n$ and at most $\frac{199}{200}n$ and density $p' = 1 - q'$ where $q' \leq q'(q, |H|/n)$.

Proof If $\kappa(G) \geq n(\log\log\log n)/(\log\log n)$, we have a large minor by Theorem 4.1 of [64]. Otherwise, we have a small cutset S, with $|S| = \kappa(G)$, and if we choose any division of G by this cutset, this induces X, Y satisfying the conditions of Corollary 5.7 (since, for any choice of X, Y, where one of X and Y might be too small, some vertex in X has degree at most $|X|$; but the bound on the minimal degrees then implies that $|X|$, $|Y| \geq \frac{1}{50}n$). The result then follows by Corollary 5.7. $\square$

We now consider graphs with small minimal degree. For a graph G, let G_ζ be the result of applying the operation 'remove a vertex of minimal degree' $\zeta|G|$ times to G, where each time the vertex removed is of degree less than $\frac{1}{50}n$.

Lemma 5.10 *Let $\epsilon > 0$ be given. Then there exist N and $\delta > 0$ such that the following assertion holds.*

Let G be a graph of order $n > N$ and edge density at least $p \geq p_0$. Suppose $\delta(G) < \frac{1}{50}n$. Let $\zeta \leq \frac{1}{50}$, and suppose that G_ζ exists. Then G_ζ has density $p' = 1 - q'$ where $q' \leq q'(q, 1-\zeta)$. Further, if $\zeta \geq \epsilon^2$, then $q' \leq q'(q, 1-\zeta, \delta)$.

Proof We use $\delta = 10^{-3}\epsilon^2$, and, for convenience, put $\delta = 0$ when considering $\zeta < \epsilon^2$.

We have $e(G_\zeta) \geq e(G) - \frac{1}{50}\zeta n^2$, so

$$\begin{aligned} p' &\geq (\tfrac{1}{2}p - \tfrac{1}{50}\zeta)/\left(\tfrac{1}{2}(1-\zeta)^2\right) \\ &\geq (1+2\zeta)(p - \tfrac{1}{25}\zeta) \\ &\geq p + 0.8\zeta \end{aligned}$$

since $p \geq p_0$. Thus $q' \leq q - 0.8\zeta$.

We want to show that $q' \leq q^{(1+\delta)^2/(1-\zeta)^2}$; so it will suffice to show that $(q-0.8\zeta)^{(1-\zeta)^2} \leq q^{(1+\delta)^2}$; that is, $q \times q^{-2\zeta+\zeta^2} \times (1-0.8\zeta/q)^{(1-\zeta)^2} \leq q \times q^{2\delta+\delta^2}$, or, equivalently, cancelling a factor of q and taking logarithms, that

$$0 > \left(\log(1/q)\right)(2\zeta - \zeta^2 + 2\delta + \delta^2) + (1 - 2\zeta + \zeta^2)\log(1 - 0.8\zeta/q).$$

We have that $\log(1/q) \leq e^{-1}/q < 0.38/q$, and $\log(1 - 0.8\zeta/q) \leq -0.8\zeta/q$, so it will suffice to show that

$$\begin{aligned} 0 &> (0.38/q)(2\zeta - \zeta^2 + 2\delta + \delta^2) - (0.8\zeta/q)(1 - 2\zeta + \zeta^2) \\ &= (1/q)\left(-0.04\zeta + 1.22\zeta^2 - 0.8\zeta^3 + 0.38(2\delta + \delta^2)\right) \\ &\leq (1/q)(-0.03\zeta + 1.22\zeta^2 - 0.8\zeta^3) \end{aligned}$$

by our choice of δ. This result holds provided $\zeta \leq 0.025$. □

We now use the above results to show that general graphs of a given density have minors as large as random graphs, if the density is sufficient or a connectivity condition applies.

Lemma 5.11 *Let $\epsilon > 0$ be given. Then there exists N such that the following assertion holds.*

Let G be a graph of order $n > N$ and edge density at least p, where $0.9999 < p < 1 - \epsilon$. Then G contains a K_t minor for $t > (1-\epsilon)n/\sqrt{\log_{1/q} n}$ (where, as usual, $q = 1 - p$).

Proof Repeatedly remove the vertex of minimal degree from G, until the minimal degree is at least $\frac{1}{50}n$; say that we have removed ζn vertices. Then $\zeta < \frac{1}{50}$, and G_ζ has density $p' = 1 - q'$ where $q' \leq q'(q, 1-\zeta)$ by Lemma 5.10. Put $n' = (1-\zeta)n = |G_\zeta|$.

If $\kappa(G_\zeta) \geq n'(\log\log\log n')/(\log\log n')$, then Lemma 5.11 follows from Theorem 4.1 of [64]. So suppose that $\kappa(G_\zeta) < n'(\log\log\log n')/(\log\log n')$. Then, as in the proof of Corollary 5.9, we have a small cutset S, with $|S| = \kappa(G_\zeta)$, and if we choose any division of G_ζ by this cutset, this induces X, Y satisfying the conditions of Corollary 5.7 (since, for any choice of X, Y, where one of X and Y might be too small, some vertex in X has degree at most $|X|$; but the bound on the minimal degrees then implies that $|X|$, $|Y| \geq \frac{1}{50}n'$). However, the density condition on G means that we cannot have such X, Y. □

The next lemma shows that general graphs of a given density have minors as large as random graphs, if the density is sufficient or a connectivity condition applies.

Lemma 5.12 *Let $\epsilon > 0$ be given. Then there exists N such that the following assertion holds.*

Let G be a graph of order $n > N$ and edge density at least p, where $\epsilon < p < 1 - \epsilon$. Suppose that either $\kappa(G) \geq n(\log\log\log n)/(\log\log n)$ or $p > p_0 + \epsilon$. Then G contains a K_t minor for $t > (1-\epsilon)n/\sqrt{\log_{1/q} n}$ (where, as usual, $q = 1 - p$).

Proof The well-connected case is just Theorem 4.1 of [64]; when $p \geq 0.9999$, the result will follow by Lemma 5.11. To prove the general result, we apply a bounded number of operations to our graph, each moving from

(H', p') (where initially $(H', p') = (G, p)$) to (H'', p'') where H' is a subgraph of G of density at least p', H'' is a subgraph of G with density at least p'', where $p'' = 1 - q'(1 - p', |H''|/|H'|)$, so ensuring that at all stages $|H'|\Big/\sqrt{\log_{1/q'} |H'|} > n/\sqrt{\log_{1/q} n}$ (where $q' = 1 - p'$). These operations are of the following forms. After any two of these operations have been consecutively applied, the new graph H''' satisfies $\frac{1}{10000}|H'| \leq |H'''| \leq \frac{199}{200}|H'|$. The upper bound ensures that the number of steps is bounded, because the density must significantly increase; the lower bound ensures that p' stays bounded above by some quantity less than 1, so that Theorem 4.1 of [64] can indeed be applied.

1. If $p' \geq 0.9999$, we have our minor by Lemma 5.11.

2. If the connectivity is high, $\kappa(H') \geq |H'|(\log\log\log |H'|)/(\log\log |H'|)$, we have our minor by Theorem 4.1 of [64].

3. Otherwise, if some cutset of order $\kappa(H')$ splits the graph into parts each of which has order at least $\frac{1}{60}|H'|$, then the conditions of Corollary 5.7 apply and by Corollary 5.7 we have our (H'', p'') with $|H''| < \frac{199}{200}|H'|$.

4. Otherwise, $\delta(H') < \frac{1}{50}|H'|$. Remove successively a vertex of minimal degree until all vertices have degree at least $\frac{1}{50}|H'|$ or at least $\frac{1}{50}|H'|$ vertices have been removed, forming the subgraph $H'' = H'_{\zeta}$. In either case, Lemma 5.10 shows that this subgraph is sufficiently dense. This is the only operation that might not significantly reduce $|H'|$; but if it does not; then $\delta(H'') \geq \frac{1}{50}|H''|$, so the next operation must be one of the first three above.

The number of passes through the above loop is bounded, so eventually one of the first two operations listed applies and we have our minor. □

Given this result, we can now prove Theorem 5.1.

Proof of Theorem 5.1 Suppose ϵ small. If

$$\kappa(G) \geq n(\log\log\log n)/(\log\log n),$$

the result follows from Lemma 5.5. Otherwise, repeatedly remove a vertex of minimal degree from G, until either the minimal degree is at least $\frac{1}{50}n$ or $\epsilon^2 n$ vertices have been removed.

If $\epsilon^2 n$ vertices have been removed, then by Lemma 5.10 the resulting graph is H', with density at least $p' = 1 - q'$, where $q' = q'(q, 1-\epsilon^2, \delta)$. By Lemma 5.12 applied to H', q' and $\delta/2$, $H' \succ K_t$ where $t \geq (1 - \delta/2)|H'|\Big/\sqrt{\log_{1/q'}|H'|}$; by the definition of $q'(q, 1-\epsilon^2, \delta)$ we have $t \geq (1 - \delta/2)(1+\delta)n/\sqrt{\log_{1/q} n} > (1+\delta/6)n/\sqrt{\log_{1/q} n}$, as required.

Otherwise, say ζn vertices were removed, where $\zeta < \epsilon^2$. The numbers removed from X and Y may not be equal, so remove a few more vertices until they are, yielding a subgraph H'; so no more than $2\epsilon^2 n$ vertices are removed in total. For H' of density p', we have that at least one of $|p'_X - p'|$, $|p'_{XY} - p'|$, $|p'_Y - p'|$ exceeds $\epsilon/2$, and $p_0 + \epsilon/2 < p' < 1 - \epsilon/2$. If $\kappa(H') \geq |H'|(\log\log\log|H'|)/(\log\log|H'|)$, the result again follows from Lemma 5.5 applied to H' and $\epsilon/2$.

Otherwise, apply Corollary 5.8 to H' and $\epsilon/2$. Either it gives the required minor, or it reduces to a subgraph H'' of density $p'' = 1 - q''$ where $q'' \leq q'(q', |H''|/|H'|, \delta)$. Now Lemma 5.12 applied to H'', q'' and $\delta/2$ gives the result, as before. $\square$

5.5 Structure of extremal graphs

We are now in a position to give a description of the extremal graphs with no K_t minor, as was attempted by Thomason [64]; as with the argument of Thomason, this is an outline, but the details differ. First consider those extremal graphs G that are minor-minimal in $\mathcal{G}_{d,k}$, the class defined in Section 3.5, where d is slightly less than $c(t)$ (the extremal function) and $k = \lceil d/\log\log\log d \rceil$. Then G has sufficient connectivity that the arguments for sparse graphs (as given in Section 3.5) imply that G has a large complete minor if it is too large, and if G is not too large, then as in the proof of Theorem 3.15, for it to be extremal it must be a graph of density $1 - \lambda$ and order $t\sqrt{\log_{1/\lambda} t}$. By Theorem 5.1, G is quasi-random.

We now claim that the general extremal graph is made up of disjoint quasi-random graphs of the form described with only $o(nd/\log\log\log d)$ additional edges. If we consider an extremal graph, which must be in $\mathcal{G}_{d,k}$, then it must have a minor of the form described. This minor is obtained by a series removals of vertices of small degree, contractions of edges with a small number of triangles on them, removals of edges to reduce the number of edges to the minimum for the class, and moving to $G[C \cup S]$ where S is a small cutset and C is a component of $G - S$. When a cutset is removed and one half of the graph taken, neither half can have many edges fewer than extremal graphs, since then the other half would have too many edges. So both halves in such a case are essentially extremal, and so of the form described. If some vertex and edge deletions and edge contractions of the types described are used to arrive at such a graph from G, then consider the parts of G from which the individual quasi-random parts of the extremal graph of the described form arose. For the original graph to have been extremal, none of these parts can have a significantly greater average degree than that of G, but nor can they (at any stage of the contraction) have a significantly greater order and similar average degree. Thus only a few edges and vertices are removed in this way, and the extremal graphs are indeed made of quasi-random graphs. (Thomason describes the structure as being a tree of quasi-random graphs. There is not, however, a meaningful sense in which the structure is a tree: no pair of the quasi-random graphs can have many edges between them, and if a few edges are between the quasi-random graphs in the form of a tree, a similar number of edges could also be present in another structure in the extremal graphs.)

Chapter 6

Extremal problem for connectivity

6.1 Introduction

Thomason [64] showed that the average degree that forces a graph to have a K_t minor is $O\left(t\sqrt{\log t}\right)$. We saw in Chapter 3 how this can be extended to some other dense minors. The extremal graphs are random graphs of a certain order and constant density, or (as we saw in Chapter 5) larger graphs made up of copies of these with few edges between them.

The small extremal graphs, being random graphs, have connectivity almost surely equal to their minimum degree [3]. The larger ones, however, have much smaller connectivity—no more than $O(t)$. It is thus natural to ask, if we require the graph G to be sufficiently large in terms of t, what connectivity will force a K_t minor. In particular, we might hope for a bound linear in t. Equivalently, given a sufficiently large t-connected graph, we may ask how large a complete minor it must have.

This question has some relation to questions of linking. This concept is defined as follows:

Definition 6.1 *A graph G is said to be k-*linked *if $|G| \geq 2k$ and, for all distinct vertices $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$ of G, there exist vertex-disjoint paths from x_i to y_i for all i.*

Thomassen [65] characterises non-2-linked graphs, a characterisation we will use later in this chapter; this characterisation is also given by Seymour [58]. A connectivity of 6 implies that a graph is 2-linked. The exact connectivity that implies that a graph is k-linked is not known for $k \geq 3$, but Bollobás and Thomason [4] showed that it is at most $22k$. Robertson and Seymour [54] showed that a $2k$-connected graph with a K_{3k} minor is k-linked. Thus, if sufficiently large t-connected graphs must have K_u minors for $u > \frac{3}{22}t$, this would yield an improvement on the bound of [4] for sufficiently large graphs.

In this chapter, we do not find a general bound on the Hadwiger number of a large t-connected graph. However, in one specific case—where the graph has a long chain of cutsets of size t—we find there must be a K_u minor for $u = t - 4$ except if the graph satisfies some restrictive conditions, as described at the end of the chapter, and for $u = \lfloor t/4 \rfloor$ except possibly if the graph satisfies further restrictive conditions.

On the other hand, we can look for arbitrarily large t-connected graphs without large complete minors. A 1-connected graph has a K_2 minor; a 2-connected graph has a K_3 minor; a 3-connected graph has a K_4 minor. The examples $K_t + \overline{K_n}$ provide arbitrarily large t-connected graphs with K_{t+1} minors but no larger minors. The icosahedron is 5-connected, but planar so has no K_5 minor; the faces of the icosahedron may be subdivided into smaller triangles to yield arbitrarily large 5-connected graphs with no

K_5 minor. More generally, if I is the graph of the icosahedron, we may put $I_t = I + K_{t-5}$ for $t \geq 5$, which is a t-connected graph with no K_t minor, and then $I_t + \overline{K_n}$ is $(t+1)$-connected with no K_{t+1} minor.

6.2 Long chains of cutsets

The only case we study in this chapter is where the graph G has a long chain of cutsets of order t, where $\kappa(G) = t$ and t is odd. The following defines the notion of a chain of cutsets; 'long' here means sufficiently long for the subsequent arguments to work, which requires a length of at least $\Omega\left(2^{\binom{t}{2}}\right)$.

Definition 6.2 *A graph G of connectivity t has a* chain of cutsets of length k *if the vertices of G can be partitioned into disjoint sets $S_1, S_2, \ldots, S_k, T_1, T_2, \ldots, T_k, W_{1,2}, W_{2,3}, \ldots, W_{k-1,k}$ (some of which may be empty) with the following properties:*

- *$|S_i| = t$ for all i and each S_i is a cutset. (We refer to the S_i as the chain of cutsets.)*
- *Removing S_i separates the following sets of vertices from each other: $S_1 \cup \cdots \cup S_{i-1} \cup T_1 \cup \cdots \cup T_{i-1} \cup W_{1,2} \cdots \cup W_{i-1,i}$; T_i; and $S_{i+1} \cup \cdots \cup S_k \cup T_{i+1} \cup \cdots \cup T_k \cup W_{i,i+1} \cup \cdots \cup W_{k-1,k}$. Some of these sets may be empty, and some of them may themselves be disconnected.*

Thus, the set $W_{i,i+1}$ consists of those vertices 'between' S_i and S_{i+1} in the chain; the set T_i consists of those vertices 'hanging off the side of the chain' at S_i. The sets T_i do not play a significant rôle in the following arguments.

The connectivity implies that, by Menger's theorem [38], we can find t paths from S_i to S_{i+1} within $W_{i,i+1}$. Joining these paths gives rise to long paths throughout the chain. Such long paths will form the components of our minor; if there is an edge from one path to another within $T_i \cup S_i \cup W_{i,i+1} \cup S_{i+1} \cup T_{i+1}$, or a path joining the two paths within those vertices (excluding those vertices already in the other paths), that gives an edge of the minor. (We cannot in general take multiple edges of the minor from the same $T_i \cup S_i \cup W_{i,i+1} \cup S_{i+1} \cup T_{i+1}$, since the paths between the components of our minor might cross. However, this is not a problem since we are assuming that our chain of cutsets is very long; it suffices to show that any single edge can be found within a chain of cutsets of bounded length.)

If we were trying to find a K_t minor, that would give too little room to manoeuvre; instead we will find a smaller minor; the smaller the minor to be found, the stronger the additional conditions that must be satisfied by a graph without that minor. Suppose we fix some choice of the paths between the S_i, and put $S_i = \{ s_{i,j} : 1 \leq j \leq t \}$, where the paths from S_i to S_{i+1} go from $s_{i,j}$ to $s_{i+1,j}$ for all j; let this path (including its endpoints) be $P_{i,i+1;j}$. Suppose now we wish to find a K_r minor for some $r < t$. Suppose that we have partially assembled components of the minor (but not all pairs yet have edges between them) using the chain up to S_i; within S_i, each part contains a distinct vertex of S_i. We could extend the paths to S_{i+1} using the paths $P_{i,i+1,j}$; and we might gain some edges of our minor this way. But if s_{i,j_1} is in one part of our minor, and s_{i,j_2} is not in our minor, and there is some path between the paths $P_{i,i+1;j_1}$ and $P_{i,i+1;j_2}$ within $T_i \cup S_i \cup W_{i,i+1} \cup S_{i+1} \cup T_{i+1}$ that does not intersect the other paths, we could instead put both those two paths, and the path between them, in the component of the minor that contains s_{i,j_1}. We might then just follow the given paths from S_{i+1} to S_{i+2}; so, in S_i the vertex s_{i,j_1} is present in our minor, but the vertex s_{i,j_2} is not, whereas in S_{i+2} the vertex s_{i+1,j_2} is present in our minor but the vertex s_{i+2,j_1} is not.

This method means that we can consider much of the problem of finding minors in terms of a simpler problem of varying which vertices of each S_i are present in which parts of our minor. If $i_1 < i_2$, define H_{i_1,i_2} to be a graph with vertex set $\{1, 2, \ldots, t\}$ which has an edge from p to q if and only if there is a path between $P_{i_1,i_1+1;p} \cup \cdots \cup P_{i_2-1,i_2;p}$ and $P_{i_1,i_1+1;q} \cup \cdots \cup P_{i_2-1,i_2;q}$ within $T_{i_1} \cup S_{i_1} \cup W_{i_1,i_1+1} \cup \cdots \cup W_{i_2-1,i_2} \cup S_{i_2} \cup T_{i_2}$ that does not intersect any of the other paths. If we enter S_{i_1} with r parts of our minor, A_i for $1 \leq i \leq r$, where $s_{i_1,a(i)}$ is the unique element of S_{i_1} in A_i at that time, we may then enter S_{i_2+1} with expanded parts A'_i of our minor, where $s_{i_2+1,a'(i)}$ is the unique element of S_{i_2+1} in A'_i, where $a(i) = a'(i)$ for all i except one; for that one i, $a'(i)$ being some value that is no $a(j)$ where there is an edge from $a(i)$ to $a'(i)$ in H_{i_1,i_2}.

Equivalently, the problem can be considered as one of moving numbered counters on the vertices of the H_{i_1,i_2}; at S_{i_1} we have some arrangement of r distinct counters (corresponding to the parts of the minor) on the set $V(H) = \{1, 2, \ldots, t\}$, and at S_{i_2} one counter has been moved to a vertex that neighbours it in H_{i_1,i_2}. We wish to find such a sequence of moves on disjoint $[i_1, i_2]$ intervals that eventually leads every counter to neighbour every other counter during some interval not used for moving counters. Now, the chain of cutsets is long, so some (labelled) graph H must appear many times as graphs H_{i_1,i_2}. We need only consider those graphs that appear many times. We will see that a large class of them do allow the counters to be permuted in the required manner, yielding our minor. Then graphs H in that class can only appear a few times, so we can consider a long subchain in which no such graphs appear at all.

In order to work with moving counters on these graphs, we need that the graphs H are connected; this is the reason for the requirement that t is odd. For, we claim that $H_{i,i+2}$ is connected; if not, it has some connected component X of order less than $t/2$. But if we then remove the vertices $s_{i,x}$ and $s_{i+2,x}$ for $x \in X$ from G, connectivity means that G must still be connected, but the definition of $H_{i,i+2}$ means that the vertices $s_{i+1,x}$ for $x \in X$ have been disconnected from the vertices $s_{i+1,y}$ for $y \notin X$. Thus we have a contradiction, so $H_{i,i+2}$ must have been connected.

We now consider the problem where we have some connected graph H of order t, $r = t - s$ numbered counters on some subset of the vertices of H, and repeatedly move a single counter to an adjacent unoccupied vertex. If such moves can yield any given arrangement of the counters and unoccupied vertices, then we have our minor; we shall see that graphs H for which some arrangements cannot be reached have a fairly restricted structure. It is easy to see that if the contents of any pair of adjacent vertices can be transposed (with the other vertices unaffected), combinations of those transpositions will yield all permutations. Trivially a counter can be transposed with an adjacent unoccupied vertex, so it suffices to show that the counters on any pair of adjacent occupied vertices can be transposed. Say we wish to transpose adjacent counters 1 and 2; in many cases we will achieve this by conjugation: make some series of moves so that the two counters are conveniently placed to transpose them by a short sequence of moves; make that short sequence; and make the reverse of the first sequence, restoring the original position except with counters 1 and 2 transposed.

The first lemma gives just such a local configuration allowing the transposition of counters.

Lemma 6.1 *If H has a star subgraph in which at least 2 vertices are unoccupied, then any permutation of the counters in that subgraph may be achieved.*

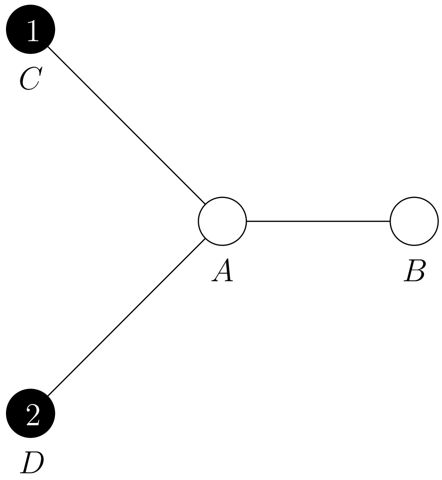

Figure 6.1: Moving counters in a star configuration

Proof If there are fewer than 4 vertices in the star this is trivial. Suppose there are at least 4 vertices in the star. If the centre of the star is not vacant, move the counter from there to a vacant vertex of the star (this move can later be reversed, a conjugation as described above). Then if A is the centre of the star (vacant), B is another vacant vertex of the star, C is a vertex of the star occupied by counter 1 and D is a vertex of the star occupied by counter 2, as shown in Figure 6.1, we may swap counters 1 and 2 by moving 1 from C to A then to B; counter 2 from D to A then to C, and counter 1 from B to A then to D. □

We next aim to show that the contents of 2-edge-connected subgraphs with a few unoccupied vertices can be arbitrarily permuted. First we consider cases of a single cycle with an additional vertex or edge.

Lemma 6.2 *If H has a cycle and some vertex of that cycle is adjacent to some vertex of H not in the cycle, and either there are at least 2 unoccupied vertices in the cycle or there is at least one unoccupied vertex in the cycle*

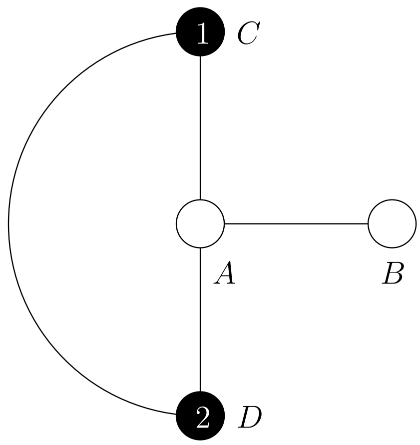

Figure 6.2: Moving counters in a cycle with adjacent vertex

and the vertex outside the cycle to which some vertex in the cycle is adjacent is also unoccupied, any permutation of the counters in that cycle may be achieved.

Proof We may clearly move counters round the cycle to place the unoccupied vertices in any desired positions on the cycle. If there are 2 unoccupied vertices in the cycle, this means we may place one of them next to the outside vertex adjacent to the cycle, then move any counter on that vertex into the cycle. Thus we need only consider the case where the vertex adjacent to the cycle is unoccupied. We may arrange for any pair of counters that are adjacent in the cycle (ignoring unoccupied vertices) to be on either side of an unoccupied vertex which is adjacent to the outside vertex, as shown in Figure 6.2. As in Lemma 6.1, those two counters may be transposed, and transpositions of adjacent pairs of counters can achieve all permutations. □

Lemma 6.3 *If H has a cycle with a chord (i.e., an edge not in the cycle that joins one vertex of the cycle to another vertex of the cycle) and at least*

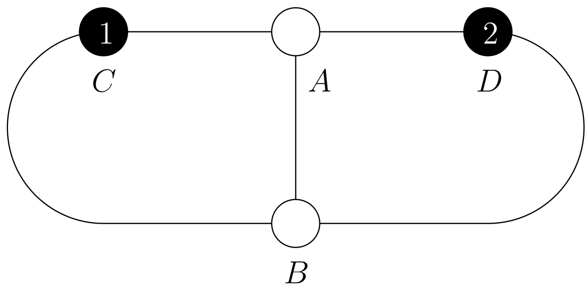

Figure 6.3: Moving counters in a cycle with a chord

2 *vertices of the cycle are unoccupied, any permutation of the counters in that cycle may be achieved.*

Proof To transpose any given pair of counters that are adjacent in the cycle (ignoring unoccupied vertices), move counters round the cycle so that the vertices at either end of the chord are unoccupied and the two counters to be transposed are on either side of one end of the chord, as shown in Figure 6.3. Lemma 6.1 then applies to transpose those counters. □

We now derive a more general result for 2-connected subgraphs.

Lemma 6.4 *If H is not a cycle, and has a* 2*-connected subgraph in which at least* 2 *vertices are unoccupied, any permutation of the counters in that subgraph may be achieved.*

Proof It suffices to show that any pair of counters in the 2-connected subgraph may be transposed, so consider some such pair of counters; 2-connectivity implies that they both lie in some cycle. If they lie in a cycle with at least 2 unoccupied vertices, then Lemma 6.2 or Lemma 6.3 implies that the counters in that cycle may be arbitrarily permuted. (The requirement that H is not a cycle is needed to avoid the case of the 2-connected subgraph being a cycle, and there being no additional edges in H to any vertex of that cycle.) If they lie in a cycle with just 1 unoccupied vertex, then the 2-connected subgraph has some other unoccupied vertex, which must lie on a path across the cycle (since all vertices of the 2-connected subgraph not in the cycle must lie on some path across it). Now counters may be moved on that path so that the unoccupied vertex lies next to the cycle, and Lemma 6.2 applies.

It thus remains only to consider the case where no cycle (in our subgraph) on which both the counters to be swapped lie has any unoccupied vertices. But now consider some unoccupied vertex in our subgraph; it lies on a path across the cycle, and, since no cycle contains both our given counters and an unoccupied vertex, at least one of the vertices where the path meets the cycle is not occupied by one of those given counters. So we may move the counter on that vertex into the path, causing there to be an unoccupied vertex on our cycle, a case in which we have seen that the counters can be swapped. □

A 2-edge-connected graph is a tree of 2-connected graphs sharing single vertices. Applying the above lemmas to those 2-connected subgraphs allows the following result for 2-edge-connected graphs to be found.

Lemma 6.5 *If H is not a cycle, and has a* 2*-edge-connected subgraph in which at least* 3 *vertices are unoccupied, any permutation of the counters in that subgraph may be achieved.*

Proof We prove this by induction on the number of 2-connected components in the 2-edge-connected subgraph. It holds where there is just one such component by Lemma 6.4, so it will suffice to suppose that our 2-edge-connected subgraph is the union of two 2-edge-connected subgraphs, sharing a single vertex, such that the result hold for each of those smaller 2-edge-connected subgraphs. By conjugation, it will suffice to suppose that the unoccupied vertices are the common vertex and one vertex adjacent to it in each smaller subgraph, and show that any permutation with those being the unoccupied vertices can be achieved. We may also assume that there are exactly 3 unoccupied vertices.

Let U and V be the 2-edge-connected subgraphs, v be the common vertex, c be some counter on a neighbour of v in U and d be some counter on a neighbour of v in V. We claim that any permutation within U can be achieved; any permutation within V can be achieved; and we can transpose c and d. This clearly suffices to prove the result.

Certainly c and d can be transposed, since we have a star as in Lemma 6.1. We will show that any permutation within U can be achieved, and the corresponding result for V will follow in exactly the same way. If c is the only counter in U, then the result is trivial. Otherwise, by moving c into V via v, we may achieve any permutation of all the other counters of U. But we may also transpose c with any other counter of U: move that counter into a cycle with c, transpose within that cycle by Lemma 6.2, and reverse the original move of the other counter into that cycle. Thus indeed we achieve any permutation of the counters within U, and so of those in the whole 2-edge-connected subgraph. □

A general connected graph H is a tree of disjoint 2-edge-connected graphs joined by trees. The following result gives circumstances under which counters on such a graph may be permuted.

Lemma 6.6 *Let H be a connected graph of order t which is not 2-edge-connected. Let P be the longest path in H with the properties that no edge* *of P lies in a cycle and that $d_H(v) = 2$ for all vertices v of P except the endpoints of P. Suppose there are $r \leq t-3$ counters on H. If P has no more than $t-r-1$ vertices, then the counters may be permuted.*

Proof Let u and v be two adjacent vertices with counters a and b on them. It will suffice to show that those two counters can be transposed.

First suppose that uv does not lie in a cycle; suppose it lies in a path Q with $d_H(w) = 2$ for all vertices w of Q except for the endpoints of Q, where $d(v) \neq 2$. Write $|Q| = k \leq t-r-1$, so there are at least $k+1$ unoccupied vertices of H. The vertices of H may be divided into those to the left of u (i.e., those in components other than that containing v after the vertex u is removed) and those to the right of v (defined likewise). If there are ℓ unoccupied vertices to the left of u and r unoccupied vertices to the right of v, then $\ell + r \geq k+1$. Moving counters appropriately in the part of the graph to the left of u, and the part to the right of v, this means that even after moving out of Q all counters apart from u and v that can be moved out of Q, there are at least $k+1-(k-2) = 3$ unoccupied vertices outside of Q, so at least 2 such vertices in one half of the graph (excluding Q), say the half to the left of u. Move a and b into that half. If they can go in different branches off the endpoint of the path in that half, then put them in different branches and apply Lemma 6.1 to transpose them; if not, having placed them in the same branch then move one counter from another branch into the path, and then again we may easily transpose them.

Now suppose that uv lies in a cycle. Consider the maximal 2-edge-connected subgraph in which uv lies. If there are 3 unoccupied vertices in that subgraph, we are done by Lemma 6.5. If not, consider where the unoccupied vertices outside of that subgraph are; if the parts of the graph they lie in are joined to our subgraph at occupied vertices other than u and v, they may be moved into our subgraph, and again we are done. So all but at most 2 unoccupied vertices are outside our subgraph, in parts of the graph joined to our subgraph either at unoccupied vertices (to which we may suppose no counter of our subgraph may be moved without disturbing a or b) or at u or v.

If now there is an unoccupied vertex outside our subgraph adjacent to u, move a to that vertex, then b to u; then a and b are adjacent on an edge that is not part of a cycle and the previous part of this proof applies; likewise if there is an unoccupied vertex outside our subgraph adjacent to v. So all outside unoccupied vertices are in parts adjacent to unoccupied vertices of our subgraph.

By some movements of the counters on our subgraph, some counter on that subgraph can be moved out. Consider the walk by which it moves out; the counter on the last unoccupied vertex of that walk could be moved out without disturbing other vertices, so that counter must be u or v, say u. But now v could follow u out along that walk, and as before we now have them adjacent and not in a cycle. □

In particular, let $r = \lfloor t/4 \rfloor$. In order not to be able to permute all the counters, either H must be a cycle, or it must contain a path with at least $t - r \geq 3t/4 \geq 3r$ vertices that is not part of a cycle and has no branches except at its ends.

We return now to the original problem of finding minors, where the minors we will find are K_r where $r = \lfloor t/4 \rfloor$ and t is sufficiently large, or (in some cases) $r = t - 4$. Only the graphs H just described can occur many times as graphs H_{i_1,i_2} for $i_2 \geq i_1 + 2$. If a cycle occurs many times, the only other graphs that can occur many times are paths that are that cycle minus one edge, since any edge between long paths in G not adjacent in the cycle would allow paths P_i to be swapped, so we have a long chain of cutsets where only the cycle and such subpaths occur; in this region, the vertices of each $T_i \cup W_{i,i+1} \cup T_{i+1}$ can be divided into those of each path, those between each pair of adjacent paths, and those hanging off a single path. If a cycle does not occur many times, consider any graph H that does. In the path in H with at least $3r$ vertices, there may be found a 'central' region of $2r$ vertices, such that there are r vertices to either side of that region in H. Having r vertices to either side means that the long paths in G can be moved out of that region as necessary. Thus no graph H_{i_1,i_2} with any edge in that central region (or between that central region and the rest of the graph) other than those of the path in H can occur many times (otherwise arbitrary paths could be transposed), so there is a long chain of cutsets in G, within which the vertices of $T_i \cup W_{i,i+1} \cup T_{i+1}$ can be divided into the paths of the central region, the vertices to each side of it, the vertices between each pair of adjacent paths in the central region, and those hanging off a single path of the central region. Where H is a cycle we will find a K_r minor; otherwise, if G does not have such a minor then it must have the form just described and large parts of G must have the structure given in Lemma 6.8.

A graph G is said to be (x_1, x_2, y_1, y_2)-*linked* if there exist vertex-disjoint paths in G from x_1 to y_1 and from x_2 to y_2. For example, if x_1 and x_2 are endpoints of paths in S_i, and y_2 and y_1 are endpoints of those respective paths in S_j, and the region of G between S_i and S_j bounded by those paths is (x_1, x_2, y_1, y_2)-linked, then the paths may be swapped. This means that if G does not have a K_r minor then various regions of G must not be (x_1, x_2, y_1, y_2)-linked.

Thomassen [65] characterised non-(x_1, x_2, y_1, y_2)-linked graphs as follows:

Theorem 6.7 (Thomassen [65]) *Suppose G is not (x_1, x_2, y_1, y_2)-linked. Then G is a subgraph on the same vertex set of a graph G' of the following form:*

G' is a planar graph, with unbounded face the 4-cycle $x_1x_2y_1y_2$ and with the other faces forming a triangulation of that 4-cycle, in which all 3-cycles are facial cycles, with the addition for each 3-cycle S of a possibly empty complete graph K^S, disjoint from the planar graph and the other such complete graphs, joined to all the vertices of S.

Suppose some part of our graph G (which we recall has connectivity t) is not (x_1, x_2, y_1, y_2)-linked. Then it must have the above form. If any of the complete graphs K^S (or subgraphs of them) are present, this makes the part of G at most 3-connected; so any such K^S must intersect the boundary of our part of G. We will show that they do not go too far inside the boundary, so we have a smaller region of G which is planar, and so has average degree less than 6. Such regions may be joined up to cover all of H if it is a cycle, or the central part of H if it is not. In the former case, we will find that a large part of G with small boundary has average degree less than 12, a contradiction if $t > 12$. In the latter case, we would also need to consider the regions to the left and right to conclude the required minor to be present; I conjecture that in that case indeed there must be a K_r minor if t is sufficiently large, but do not have an argument to prove anything beyond the information about the structure of the central region of G given below.

The following result provides a basic planar region of G.

Lemma 6.8 *Let $t > 12$. Consider a part of the long chain of cutsets described above, containing at least 20 of the cutsets (including all the $W_{i,i+1}$ between those cutsets, and all the T_i attached to them). Consider, within that* *part of the chain,* 6 *successive long paths* P_1, P_2, P_3, P_4, P_5, P_6 *in the central part of* H *as described above. Let the path* P_i *start at* p_i *and end at* q_i*. Suppose that the region bounded by* P_1 *and* P_6 *is not* (p_1, q_6, p_6, q_1)*-linked. Then the region bounded by* P_3, P_4 *and the first and last cutsets left after removing the first and last* 9 *cutsets from the part of chain given (this region being taken to include any parts of the graph attached only to* P_3*, only to* P_4*, or being part of the* T_i *at either end attached only to the relevant endpoints) is planar.*

Proof The larger region has the structure of Theorem 6.7, so to prove that the smaller region is planar it suffices to prove that none of the K^S intersect the smaller region.

Consider one of the K^S. It intersects the boundary of the larger region; either P_1 or P_6 (but not the endpoint of either path, since those endpoints are the vertices of the 4-cycle in the given form of the region), or the endpoints of the other paths.

If it intersects P_1, say, then two of the three points that cut off that K^S are in P_1; thus it cannot also intersect P_6 (since then two of the points cutting off that K^S would also be in P_6). We have more than 3 vertex-disjoint paths from P_6 to P_1 (by t-connectivity); all of these pass through P_2, so our K^S cannot include the whole of P_2 or of any other path between P_1 and P_6 (since it can be separated from the rest of the graph by 3 vertices). Thus it may intersect P_2, but cannot reach P_3, so no part of it is in the smaller region; and this likewise holds if it intersects P_6 rather than P_1.

Now suppose it contains some endpoints of some paths; but not, as discussed above, both endpoints of any path. Say it contains the upper endpoint of some path. There is a path from P_6 to P_1 within the subregion of the upper three cutsets, passing through all intermediate paths; so the whole of the K^S is within that region, or the path from P_6 to P_1 intersects the K^S and so one of its 3 boundary points is within that region. Considering likewise the next two triples of consecutive cutsets, we see that the K^S cannot extend beyond them, so indeed the smaller region describes intersects no K^S and is planar. □

This yields our result for the case where H is a cycle.

Theorem 6.9 *Given $t > 12$ there exists $p(t)$ such that the following assertion holds.*

Suppose that the graph H, a cycle, appears at least $p(t)$ times as $H_{3i,3i+2}$ in our chain of cutsets. Then $K_{t-4} \prec G$.

Proof Suppose we do not have our minor. Given $t-4$ long paths, we may consider a region with 2 paths present surrounding 4 paths absent. Then by Lemma 6.8 the region between the central pair of paths there is planar; the choice of the central pair there was arbitrary, so we have a long region in which the subregion between each pair of consecutive paths is planar. Now each vertex gets contributions to its degree from at most 2 of those regions, so the average degree in the overall long part of the graph is less than 12. But this region is long, and has a boundary of only $2t$ vertices, so some interior vertex has a degree of at most 12, contradicting the connectivity. □

This result and Lemma 6.5 together imply that if $t > 12$ and any 2-edge-connected graph H occurs many times then $K_{t-4} \prec G$. Thus if $K_{t-4} \not\prec G$ then any graph H that occurs many times has a bridge.

Part II

Directed Graphs

Chapter 7

Introduction to directed graphs

In this part of this dissertation, we consider some extremal problems relating to directed graphs that are simple to state but surprisingly difficult to solve.

Caccetta and Häggkvist [6] made a conjecture which includes, as one case, that an oriented graph on n vertices with all vertices of out-degree at least $n/3$ must have a directed triangle. Various bounds on the minimum out-degree required have been found, but the conjecture remains unproved, even if we make the stronger requirement that both in-degrees and out-degrees must be at least $n/3$. In Chapter 8, we observe that a natural approach to the problem leads to another simple conjecture, which also seems difficult to prove, but does not seem to have been previously discussed.

A closely related problem is the conjecture of Seymour that every oriented graph contains a vertex with a *large second neighbourhood*. The *second neighbourhood* of a vertex x is the set of all out-neighbours of out-neighbours of x, that are not themselves out-neighbours of x, and x has a large second neighbourhood if its second neighbourhood is at least as large as the set of its out-neighbours. This conjecture clearly implies that, if every vertex of an oriented graph has in-degree and out-degree at least $n/3$, then the graph contains a directed triangle. The specific case for tournaments, known as Dean's conjecture [12], has recently been proved [19, 23].

The classic result of Erdős and Szekeres [17] is that in any sequence of k^2+1 integers (or reals, etc.) there is a monotone subsequence of length $k+1$. Many variations on this problem have been considered [60], but apparently not the following one: in a sequence of length $n \geq k^2 + 1$, what is the minimum number of monotone subsequences of length $k+1$? We consider this problem in Chapter 9. We give a conjecture as to the answer to this problem, supported by computational evidence, and a conjectured characterisation of all the extremal sequences for n sufficiently large in terms of k, which we prove correct under certain assumptions.

This problem is equivalent to a problem on tournaments (which shows more of the natural symmetry of the problem): given two transitive tournaments on the same n vertices, what is the minimum number of subgraphs K_{k+1} on which the tournaments entirely agree or entirely disagree? This formulation of the problem, together with standard results on the number of monochromatic triangles in a 2-coloured complete graph [20, 34], yields a proof of the correctness and completeness of the characterisation of the extremal sequences for $k = 2$ and all n.

Chapter 8

Directed triangles

This short chapter has no results but we give a new conjecture (Conjecture 8.5) related to some unsolved problems that have been previously posed.

Caccetta and Häggkvist [6] made the following conjecture:

Conjecture 8.1 (Caccetta and Häggkvist [6]) *Any digraph on n vertices, all of which have out-degree at least r, contains a directed cycle of length at most $\lceil n/r \rceil$.*

Despite the simplicity of this conjecture, it remains unproved apart from some particular cases. One simple case which remains unproved is where $r = n/3$:

Conjecture 8.2 *Any oriented graph on n vertices, all of which have out-degree at least $n/3$, contains a directed triangle.*

Even the following weaker conjecture remains unproved:

Conjecture 8.3 *Any oriented graph on n vertices, all of which have in-degree and out-degree at least $n/3$, contains a directed triangle.*

Some bounds (not as good as $n/3$) have been found on the minimum out-degree (or in-degree and out-degree) required to force a directed triangle, by Caccetta and Häggkvist [6], by de Graaf, Schrijver and Seymour [11], by Bondy [5] and by Shen [59]. (It is clear that $n/3$ is best possible; consider placing the vertices in a cyclic order, with edges from each vertex to the next $\lceil n/3 \rceil - 1$ vertices.)

Define the *second neighbourhood* of a vertex x in an oriented graph to be

$$\Gamma^{++}(x) = \left(\bigcup_{y \in \Gamma^+(x)} \Gamma^+(y) \right) \setminus \Gamma^+(x).$$

The following conjecture is due to Seymour:

Conjecture 8.4 (Seymour) *In any oriented graph, there is a vertex x with $|\Gamma^{++}(x)| \geq |\Gamma^+(x)|$. (Such a vertex is said to have a* large second neighbourhood*.)*

It is clear that Conjecture 8.4 implies Conjecture 8.3. The specific case of Conjecture 8.4 for tournaments, known as Dean's conjecture [12], has been proved by Fisher [19] and then more simply by Havet and Thomassé [23], but the general case remains open.

One natural way to attack Conjecture 8.2 is by induction. Suppose that G is a minimum counterexample, say of order n with all out-degrees equal to $\lceil n/3 \rceil$. If $n = 3k + 1$ then removing a vertex reduces out-degrees by at most 1; as $\lceil n/3 \rceil - 1 = \lceil (n-1)/3 \rceil$, G is not in fact minimal. If $n = 3k$ this approach does not lead to any simple arguments. If $n = 3k + 2$, then minimality of G implies that removing any two vertices reduces some out-degree by more than 1. Thus every two vertices must have a common in-neighbour. This leads to the following new conjecture:

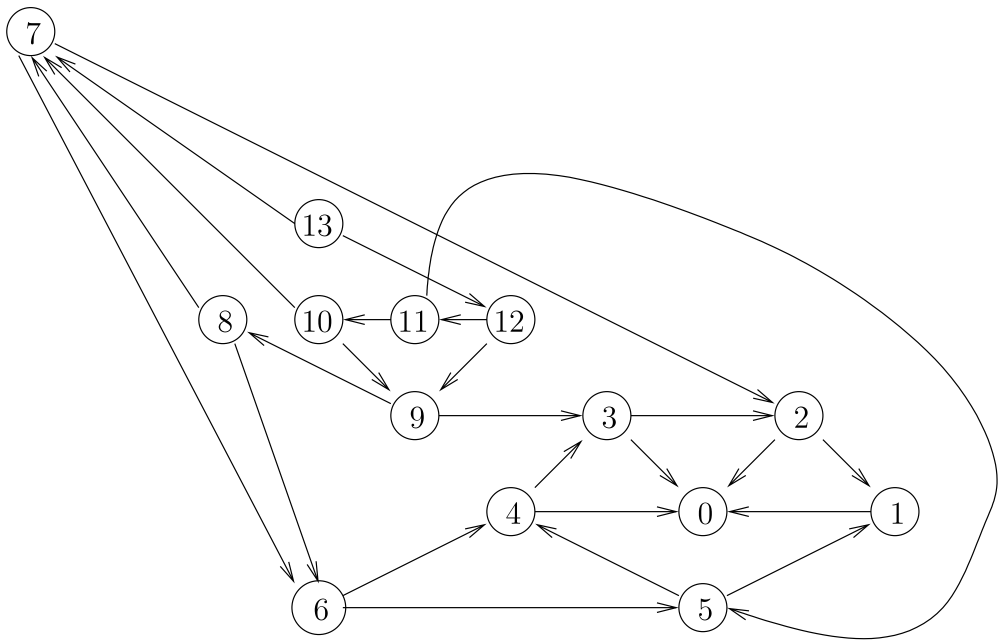

Figure 8.1: Graph with 14 vertices forced by a single edge

Conjecture 8.5 *In any oriented graph (of order at least* 2*) in which every two vertices have a common in-neighbour, there is a directed triangle.*

An example of an oriented graph in which every two vertices have a common in-neighbour is a tournament on 7 vertices, where each vertex has edges to those that are 1, 2 or 4 after it in a cyclic order.

Suppose we consider a hypothetical oriented graph in which every two vertices have a common in-neighbour, but which does not contain a directed triangle. Consider some edge in this graph, say $1 \to 0$. 0 and 1 have a common in-neighbour, which must be some other vertex, say 2. 2 and 0 have a common in-neighbour, which must be some other vertex, say 3. This argument may be repeated for appropriately chosen pairs of vertices x and y, such that no z in the part of the graph already found could be a common in-neighbour of x and y without causing either a directed triangle or a pair of edges $u \to v$, $v \to u$ to be present. Depending on the pairs of vertices chosen, various graphs can be forced in this way; such a graph, with 14 vertices, is illustrated in Figure 8.1.

If Conjecture 8.5 is false, one could then ask whether there is any constant upper bound on the girth of an oriented graph in which every two vertices have a common in-neighbour.

Chapter 9

Monotone subsequences

9.1 Introduction

A well-known result of Erdős and Szekeres [17] may be expressed as follows:

Theorem 9.1 (Erdős and Szekeres [17]) *Let n and k be positive integers. If $n \geq k^2 + 1$, then in any permutation of the integers $\{0, 1, \ldots, n-1\}$ there is a monotone subsequence of length $k + 1$.*

This problem leads to many variations, a survey of which has been made by Steele [60]. Here we consider an extremal problem that arises as a variation; this problem was posed by Mike Atkinson, Michael Albert and Derek Holton. If $n \geq k^2 + 1$, then we know there is at least one monotone subsequence of length $k + 1$; how many such sequences must there be? We write $m_k(S)$ for the number of monotone subsequences of length $k + 1$ in the permutation S. This problem is related to a question of Erdős [14] in Ramsey theory asking for the minimum number of monochromatic K_t subgraphs in a 2-coloured K_n; Erdős's conjecture about the answer to that question (that the minimum was given by random colourings) was disproved by Thomason [62].

Some upper and lower bounds are obvious. For an upper bound, note that in a random permutation, any given subsequence of length $k+1$ is monotone with probability $2/(k+1)!$. Thus some permutation has at most

$$\frac{2}{(k+1)!}\binom{n}{k+1}$$

monotone subsequences of length $k+1$. For a lower bound, note that any subsequence of length k^2+1 must have a monotone subsequence of length $k+1$, and any sequence of length $k+1$ is in $\binom{n-k-1}{k^2-k}$ sequences of length k^2+1. Thus there are at least

$$\frac{\binom{n}{k^2+1}}{\binom{n-k-1}{k^2-k}} = \frac{1}{\binom{k^2+1}{k+1}}\binom{n}{k+1}$$

monotone subsequences of length $k+1$.

A simple example will, in fact, give a better upper bound than a random permutation; this bound is, for large k, half way (geometrically) between the upper and lower bounds just given. Consider the permutation

$$\begin{aligned}
&\lfloor n/k\rfloor-1, \lfloor n/k\rfloor-2, \dots, 0,\\
&\lfloor 2n/k\rfloor-1, \lfloor 2n/k\rfloor-2, \dots, \lfloor n/k\rfloor,\\
&\dots,\\
&n-1, n-2, \dots, \lfloor (k-1)n/k\rfloor.
\end{aligned}$$

This permutation is made up of k monotone descending subsequences, each of length $\lfloor n/k\rfloor$ or $\lceil n/k\rceil$; clearly it has no monotone ascending subsequences of length $k+1$, and any monotone descending subsequences it has of length $k+1$ must lie entirely within just one of the k monotone descending subsequences into which it is divided. Thus the number of monotone subsequences of length $k+1$ is

$$(n \bmod k)\binom{\lceil n/k \rceil}{k+1} + (k - (n \bmod k))\binom{\lfloor n/k \rfloor}{k+1} \approx \frac{1}{k^k}\binom{n}{k+1}.$$

Let this number be known as $M_k(n)$. I conjecture that this is in fact the minimum number of monotone subsequences of length $k+1$.

Conjecture 9.2 *Let n and k be positive integers. In any permutation of the integers $\{0, 1, \ldots, n-1\}$ there are at least $M_k(n)$ monotone subsequences of length $k+1$.*

It would also be interesting to know the extremal configurations. It appears from computation that the behaviour for $k = 2$ is entirely different from that for $k > 2$ (although I do not have a proof that $M_k(n)$ is the correct extremum, or that the conjectured sets of extremal configurations are complete, except for $k = 2$). For $k = 2$, n even, there are $2^{n/2}$ extremal configurations; for $k = 2$, n odd, there are 2^{n-1} extremal configurations. These configurations are described in Section 9.2. Some of these configurations have both ascending and descending monotone subsequences of length $k+1$. For $k > 2$, the extremal configurations, provided n is sufficiently large in terms of k, appear to be more restricted; it seems that no extremal configuration has both ascending and descending monotone subsequences of length $k+1$. These configurations are described in Section 9.3; it is shown that, if indeed no extremal configuration has both ascending and descending monotone subsequences of length $k+1$, the characterisation is complete and correct for $n \geq k(2k-1)$. The hard part of the main result of this section, Theorem 9.8, is that the complicated characterisation of extrema (subject to the constraint that all monotone $(k+1)$-subsequences go in the same direction) is a complete and correct characterisation of all extrema; it is straightforward to see that the constraint implies that there cannot be fewer monotone $(k+1)$-subsequences than in the given extrema. (Computation suggests that—apart from the exceptional case of $k = 3$, $n = 16$, where there are also some extremal configurations not as described—all extremal configurations do indeed satisfy the given constraint.) The number of extremal configurations (under this assumption) may be described in terms of the Catalan numbers [7, 8].

The problem may be seen to be equivalent to a problem on directed graphs as follows. Consider a permutation $p_0, p_1, \ldots, p_{n-1}$. Let A be a transitive tournament on n vertices, $v_0, v_1, \ldots, v_{n-1}$, with an edge $v_i \to v_j$ for all $i < j$. Let B be a transitive tournament on the same vertices, with an edge $v_i \to v_j$ if and only if $p_i < p_j$. Now a monotone ascending subsequence of length $k+1$ corresponds to a K_{k+1} subgraph on some subset of the same vertices, all of whose edges go in the same direction in both A and B; and a monotone descending subsequence of length $k + 1$ corresponds to a K_{k+1} subgraph on some subset of the same vertices, all of whose edges go in opposite directions in A and B. Thus the problem is equivalent to: given two transitive tournaments on the same set of n vertices, what is the minimum number of K_{k+1} subgraphs on which the edge directions of the two tournaments entirely agree or entirely disagree? Furthermore, this formulation of the problem is symmetrical in A and B. In general, the problem has the following symmetries, which appear naturally in the formulation in terms of tournaments:

- The order of the permutation may be reversed (equivalent to reversing the order on A); the new permutation is $p_{n-1}, p_{n-2}, \ldots, p_0$.
- The value of p_i may be replaced by $n - 1 - p_i$ (equivalent to reversing the order on B).

- The permutation may be replaced by the permutation $q_0, q_1, \ldots, q_{n-1}$, where $q_{p_i} = i$ (equivalent to swapping A and B). This permutation is the inverse permutation to $p_1, p_2, \ldots, p_n$.

Combinations of these operations may also be applied; the symmetry group is that of the square, the dihedral group on 8 elements.

A paper based on this chapter has been published in *The Electronic Journal of Combinatorics* as [40].

9.2 The case $k = 2$

We will see that, for $k = 2$, all permutations with a minimum number of monotone 3-sequences have the following form:

Theorem 9.3 *If $n = 1$, the extremal permutation is 0. If $n = 2$, the extremal permutations are $0, 1$ and $1, 0$. If $n > 2$, all extremal sequences have the form $L, 0, n-1, R$ or $L, n-1, 0, R$, where L and R have lengths $\lfloor n/2 \rfloor - 1$ or $\lceil n/2 \rceil - 1$ and L, R is an extremal permutation of $\{1, 2, \ldots, n-2\}$ (that is, the result of adding 1 to each element of an extremal permutation of $\{0, 1, \ldots, n-3\}$). All such permutations are extremal.*

It is clear that this yields $2^{n/2}$ extremal permutations for n even and 2^{n-1} extremal permutations for n odd. For n even, there is a simple noninductive description: if the permutation is $p_0, p_1, \ldots, p_{n-1}$, then, for $0 \le t < n/2$, we have that p_t and p_{n-1-t} take the values $(n/2) - 1 - t$ and $(n/2) + t$, in some order. Table 9.1 shows the extremal permutations for $n \le 6$.

The sequences in Theorem 9.3 all have 0 and $n - 1$ adjacent. It is easy to see that Theorem 9.3 is a correct characterisation of extremal sequences with that property.

Table 9.1: Extremal permutations for $n \le 6$

$n = 1$	0			
$n = 2$	0 1		1 0	
$n = 3$	0 2 1	1 0 2	1 2 0	2 0 1
$n = 4$	1 0 3 2	1 3 0 2	2 0 3 1	2 3 0 1
$n = 5$	1 0 4 3 2	2 0 4 1 3	2 3 0 4 1	3 0 4 1 2
	1 3 0 4 2	2 0 4 3 1	2 3 4 0 1	3 1 0 4 2
	1 3 4 0 2	2 1 0 4 3	2 4 0 1 3	3 1 4 0 2
	1 4 0 3 2	2 1 4 0 3	2 4 0 3 1	3 4 0 1 2
$n = 6$	2 1 0 5 4 3	2 4 0 5 1 3	3 1 0 5 4 2	3 4 0 5 1 2
	2 1 5 0 4 3	2 4 5 0 1 3	3 1 5 0 4 2	3 4 5 0 1 2

Lemma 9.4 *Suppose $n > 2$ and that some extremal permutation has 0 and $n-1$ adjacent. Then all extremal permutations with 0 and $n-1$ adjacent are as described in Theorem 9.3, and all such permutations are extremal.*

Proof Without loss of generality, suppose a permutation with 0 and $n-1$ adjacent is $L, 0, n-1, R$; call this permutation S. Suppose that L has length ℓ and R has length r. All monotone subsequences of length 3 in L, R are also such subsequences of S. There are no monotone subsequences of S containing both 0 and $n-1$. There are no monotone subsequences of S of the form $a, 0, b$ or $a, n-1, b$, with $a \in L$ and $b \in R$. If, however, a precedes b in L, exactly one of $a, b, 0$ and $a, b, n-1$ is monotone; likewise, if a precedes b in R, exactly one of $0, a, b$ and $n-1, a, b$ is monotone. Thus $m_2(S) = m_2(L, R) + \binom{\ell}{2} + \binom{r}{2}$. This is minimal when $|\ell - r| \le 1$. □

Consider again the relation to tournaments described in Section 9.1. Suppose we colour an edge red if the two tournaments agree on the direction of that edge, or blue if the two tournaments disagree on the direction of that edge. The problem is then to minimise the number of monochromatic triangles. (However, we cannot use any 2-colouring of K_n, only one arising from two tournaments in this manner.) Goodman [20] and Lorden [34] found that the number of monochromatic triangles depends only on the sequence of red (or blue) degrees:

Theorem 9.5 (Goodman [20] and Lorden [34]) *Let K_n be coloured in red and blue. Let $d_r(v)$ be the number of red edges from the vertex v. Then there are exactly*

$$\binom{n}{3} - \frac{1}{2}\sum_{v} d_r(v)\big(n-1-d_r(v)\big)$$

monochromatic triangles.

This theorem allows us to prove correct our characterisation of extremal configurations.

Proof of Theorem 9.3 for n even The canonical extremum from Section 9.1 is of this form, and has $M_2(n) = 2\binom{n/2}{3}$ monotone subsequences of length 3. In the coloured graph corresponding to this permutation, each vertex has red degree equal to either $\lceil (n-1)/2 \rceil$ or $\lfloor (n-1)/2 \rfloor$, so the graph minimises the number of monochromatic triangles. Thus all the permutations for n even described in Theorem 9.3 are indeed extremal. Also, in the coloured graph corresponding to an extremal permutation $p_0, p_1, \ldots, p_{n-1}$, all vertices must have red degree either $\lceil (n-1)/2 \rceil$ or $\lfloor (n-1)/2 \rfloor$; in particular, the vertices corresponding to the values 0 and $n-1$ must have such red degrees. This means that 0 and $n-1$ must each be the value of one of

$p_{(n/2)-1}$ and $p_{n/2}$, so they are adjacent, and the result follows by Lemma 9.4. □

This method does not apply quite so simply for n odd, where the graphs corresponding to extremal permutations do not minimise the number of monochromatic triangles over all colourings (that is, the colourings minimising the number of monochromatic triangles do not correspond to pairs of transitive tournaments). However, the colourings are sufficiently close to extremal that with a little more effort the method can be adapted.

Proof of Theorem 9.3 for n odd The canonical extremum from Section 9.1 is of this form, so $M_2(n)$ monotone subsequences of length 3 can be attained. We will show that this is indeed extremal, and that in all extremal permutations 0 and $n-1$ are adjacent, so that the result will then follow by Lemma 9.4.

Suppose we have some extremal permutation $p_1, p_2, \ldots, p_n$, and let $\ell(v)$ be the location of the value v; that is, $p_{\ell(v)} = v$. Let the vertex corresponding to the position $\ell(v)$ with value v also be known as v. Let $d_r(v)$ and $d_b(v)$ be the numbers of red and blue edges, respectively, from the vertex v; put $d_d(v) = \frac{1}{2}|d_r(v) - d_b(v)|$. Observe that $d_r(v)\big(n-1-d_r(v)\big) = d_r(v)d_b(v) = (\frac{n-1}{2})^2 - d_d(v)^2$, so, by Theorem 9.5, the number of monochromatic triangles then is

$$\binom{n}{3} - \frac{n(n-1)^2}{8} + \sum_v d_d(v)^2.$$

Thus, we wish to minimise $\sum_v d_d(v)^2$. In the canonical extremum this takes the value $\frac{n-1}{2}$.

Suppose $0 \le v \le (n-1)/2$. Let $L = \{\, u : \ell(u) < \ell(v) \,\}$ be the set of values to the left of v, and $R = \{\, u : \ell(u) > \ell(v) \,\}$ be the set of values to the right of v. Put further $L_r = \{\, u \in L : u < v \,\}$, $L_b = \{\, u \in L : u > v \,\}$,

$R_r = \{ u \in R : u > v \}$ and $R_b = \{ u \in R : u < v \}$. Then we have $d_r(v) = |L_r| + |R_r|$ and $d_b(v) = |L_b| + |R_b|$, so

$$d_r(v) - d_b(v) = |R_r| - |R_b| - |L_b| + |L_r| = \big(|R| - |L|\big) + 2\big(|L_r| - |R_b|\big).$$

Now

$$|R| - |L| = \big(n - 1 - \ell(v)\big) - \ell(v) = 2\left(\tfrac{n-1}{2} - \ell(v)\right),$$

and

$$\big||L_r| - |R_b|\big| \le |L_r \cup R_b| = v,$$

so $d_d(v) \ge \max\{0, |\frac{n-1}{2} - \ell(v)| - v\}$. Likewise, for $(n-1)/2 \le v \le n-1$, we have $d_d(v) \ge \max\{0, |\frac{n-1}{2} - \ell(v)| - (n-1-v)\}$. Define $r(j)$ by $r(j) = j$ for $0 \le j \le (n-1)/2$ and $r(j) = n-1-j$ for $(n-1)/2 \le j \le n-1$, so we have

$$d_d(v) \ge \max\left\{0, \left|\frac{n-1}{2} - \ell(v)\right| - r(v)\right\}.$$

For $0 \le j \le (n-1)$, put $S(j) = \{ i : |\frac{n-1}{2} - i| \le r(j) \}$. That is, $S(j)$ is the set of possible value of $\ell(j)$ for which our lower bound on $d_d(j)$ would be 0. We then have

$$d_d(v) \ge |\{ (n-1)/2 \ge j \ge r(v) : \ell(v) \notin S(j) \}| = \sum_{\substack{(n-1)/2 \ge j \ge r(v) \\ \ell(v) \notin S(j)}} 1.$$

Adding over all v and reversing the order of summation then gives

$$\sum_v d_d(v) \ge \sum_{0 \le j \le (n-1)/2} |\{ v : r(v) \le j,\, \ell(v) \notin S(j) \}|.$$

For $0 \le j < (n-1)/2$, observe that $|S(j)| = 2j+1$, whereas $|\{ v : r(v) \le j \}| = 2j+2$. Thus $\sum_v d_d(v) \ge \frac{n-1}{2}$, and equality requires that each $|\{ v : r(v) \le j, \ell(v) \notin S(j) \}|$ equals 1, for $0 \le j < \frac{n-1}{2}$. Now $\sum_v d_d(v)^2 \ge \sum_v d_d(v)$, with equality only if all terms are 0 or 1. So any extremum must have $\ell(0)$ and $\ell(n-1)$ both equal to $\frac{n-1}{2}$ or $\frac{n-1}{2} \pm 1$, with one of them equal to $\frac{n-1}{2}$. So 0 and $n-1$ are adjacent. □

9.3 The case $k > 2$

For $k > 2$, it seems that, for n sufficiently large, the permutations with a minimum number of monotone $(k+1)$-subsequences have only descending, or only ascending, monotone subsequences of that length; making this assumption, we can give a characterisation of the extremal permutations for $n \geq k(2k-1)$ (which appears to be sufficiently large, except for $k = 3$, $n = 16$, where there are also some other extremal permutations). It is easy to see that this condition is equivalent to the permutation being divisible into (at most) k disjoint monotone descending subsequences, or k disjoint monotone ascending subsequences. If it can be divided into k disjoint monotone descending subsequences, there cannot be a monotone ascending $(k+1)$-subsequence, since such a sequence would have to contain two elements from one of the k descending subsequences. Conversely, if it contains only descending subsequences of length $k+1$, it can be divided into k descending subsequences explicitly; similarly to one proof of Theorem 9.1, form these subsequences by adding each element in turn to the first of the subsequences already present it can be added to without making that subsequence nondescending, or start a new subsequence if the element is greater than the last element of all existing subsequences. Any element added is at the end of an ascending subsequence, containing one element from each sequence up to the one to which the element was added, so having $k+1$ subsequences would imply the presence of a monotone ascending subsequence of length $k+1$, a contradiction.

The form of the extremal permutations (subject to the supposition described) is somewhat more complicated than that for $k = 2$. We describe the form where all the monotone $(k+1)$-subsequences are descending; the sequences for which they are all ascending are just the reverse of those we describe. If the k subsequences are of lengths ℓ_1, ℓ_2, $\ldots$, ℓ_k (where some of the ℓ_i may be 0 if there are less than k subsequences), there are at least

$$\sum_{i=1}^{k} \binom{\ell_i}{k+1}$$

monotone subsequences of length $k+1$. For this to be minimal, convexity implies that $\lfloor n/k \rfloor \leq \ell_i \leq \lceil n/k \rceil$ for all i; in particular, there are k subsequences, and no ℓ_i is 0, for $n \geq k$. To make the ordering of the ℓ_i definite, order the k subsequences by the position of their middle element (the leftmost of two middle elements, if the sequence is of even length). There are $\binom{k}{n \bmod k}$ choices of the ℓ_i satisfying these inequalities. If they are satisfied, there are at least $M_k(n)$ monotone $(k+1)$-subsequences, and exactly that number if and only if there is no monotone descending $(k+1)$-subsequence that takes values from more than one of the k subsequences. Put $s_i = \sum_{1 \leq j \leq i} \ell_i$. For each choice of the ℓ_i, we have a canonical extremum similar to that given in Section 9.1:

$$\begin{aligned} & s_1 - 1, s_1 - 2, \ldots, 0, \\ & s_2 - 1, s_2 - 2, \ldots, s_1, \\ & \ldots, \\ & s_k - 1, s_k - 2, \ldots, s_{k-1}. \end{aligned}$$

(where $0 = s_0$ and $s_k = n$).

We will describe the extrema with the given ℓ_i, supposing $n \geq k(2k-1)$. To do so we will need some more notation. Write $C_k = \frac{1}{k+1}\binom{2k}{k}$ for the k^{th} *Catalan number*. It will then turn out that there are exactly C_k^{2k-2} extrema with the given ℓ_i. Thus, the total number of extremal sequences, subject to the constraint that all monotone $(k+1)$-subsequences go in the same direction, and subject to $n \geq k(2k-1)$, will be

$$2\binom{k}{n \bmod k} C_k^{2k-2}.$$

The extrema are closely related to the canonical extremum with the given ℓ_i. In each extremum with those ℓ_i, the $\ell_i - (2k-2)$ middle values of each of the k monotone subsequences take the same values, in the same positions, as they do in the canonical extremum; the $k-1$ values at either end of each subsequence can vary, as can their positions.

The variation is described in terms of sets $C(k,p)$ of monotone descending sequences of $k-1$ integers; $|C(k,p)| = C_k$. This set is defined as follows: $C(k,p)$ is the set of monotone descending sequences $c_1, c_2, \ldots, c_{k-1}$ of integers, $p - 2k + 3 \leq c_i \leq p$ for all i, such that if $d_1, d_2, \ldots, d_{k-1}$ is the monotone descending sequence of all integers in $[p-2k+3, p]$ that are not one of the c_i, then $c_1, c_2, \ldots, c_{k-1}, d_1, d_2, \ldots, d_{k-1}$ has no monotone descending subsequence of length $k+1$.

There are various equivalent characterisations of $C(k,p)$:

Lemma 9.6 *Define $C_1(k,p)$ to be the set of monotone descending sequences $c_1, c_2, \ldots, c_{k-1}$ of integers, such that $p-k-i+2 \leq c_i \leq p-2i+2$ for all $1 \leq i \leq k-1$. Define $C_2(k,p)$ inductively as follows. Let $C_2(2,p) = \{p-1, p\}$. For $k > 2$, let $C_2(k,p) = \{\,(c_1, c_2, \ldots, c_{k-1}) : p-k+1 \leq c_1 \leq p,\ c_2 < c_1,\ (c_2, c_3, \ldots, c_{k-1}) \in C_2(k-1, p-2)\,\}$. Then $C_1(k,p) = C_2(k,p) = C(k,p)$. Furthermore, $|C(k,p)| = C_k$.*

Proof Of these definitions, C is the one that will be relevant later in proving the characterisation of extremal permutations correct. C_1 will be seen to be a direct description of C, and C_2 will be seen to be an inductive description of C_1. C_2 allows the number of such sequences to be calculated through recurrence relations, which will yield the last part of the lemma. Observe that all these definitions clearly have the property that $C(k, p_1)$ is related to $C(k, p_2)$ simply by adding $p_1 - p_2$ to all elements of all sequences in $C(k, p_2)$.

We first show that $C_1(k, p) = C(k, p)$. First consider a sequence c_1, c_2, $\ldots$, c_{k-1} in $C_1(k, p)$, letting d_1, d_2, $\ldots$, d_{k-1} be the monotone descending sequence of all integers in $[p - 2k + 3, p]$ that are not one of the c_i. If the sequence c_1, c_2, $\ldots$, c_{k-1}, d_1, d_2, $\ldots$, d_{k-1} has a monotone descending subsequence of length $k + 1$, suppose that subsequence has t values among the c_i. The last of these is at most $p - 2t + 2$. The interval $[p - 2k + 3, p]$ contains $2k-2t-1$ values smaller than $p-2t+2$; of these, at least $k-1-t$ must be among the c_i (namely, c_{t+1}, c_{t+2}, $\ldots$, c_{k-1}), so at most $k - t$ are among the d_i, so the monotone subsequence has length at most k, a contradiction. Thus $C_1(k, p) \subset C(k, p)$. Conversely, consider a sequence c_1, c_2, $\ldots$, c_{k-1} in $C(k, p)$, and let d_i be as above. Clearly $c_i \geq p - k - i + 2$ for all i; otherwise we would have $c_{k-1} < p - 2k + 3$. If we had $c_i > p - 2i + 2$, then there would be at least $2k - 2i$ lesser values in the interval $[p - 2k + 3, p]$, of which $k - 1 - i$ are among the c_j, so at least $k - i + 1$ are among the d_j; together with c_1, c_2, $\ldots$, c_i, this yields a monotone subsequence of length at least $k + 1$, a contradiction. Thus $C(k, p) \subset C_1(k, p)$.

We now show that $C_1(k, p) = C_2(k, p)$. We do this by induction on k; it clearly holds for $k = 2$ and all p. Suppose that $C_1(k - 1, q) = C_2(k - 1, q)$ for all q. If c_1, c_2, $\ldots$, c_{k-1} is in $C_2(k, p)$, then $p - k + 1 \leq c_1 \leq p$, and, since $c_1 > c_2$ and c_2, c_3, $\ldots$, c_{k-1} is in $C_2(k - 1, p - 2) = C_1(k - 1, p - 2)$, the sequence of the c_i is descending and $(p - 2) - (k - 1) - (i - 1) + 2 = p - k - i + 2 \leq c_i \leq (p - 2) - 2(i - 1) + 2 = p - 2i + 2$ for all $2 \leq i \leq k - 1$, so the sequence is in $C_1(k, p)$. Conversely, if c_1, c_2, $\ldots$, c_{k-1} is in $C_1(k, p)$, then for $2 \leq i \leq k - 1$ we have $p - k - i + 2 = (p - 2) - (k - 1) - (i - 1) + 2 \leq$

$c_i \leq p - 2i + 2 = (p-2) - 2(i-1) + 2$, so that $c_2, c_3, \ldots, c_{k-1}$ is in $C_1(k-1, p-2) = C_2(k-1, p-2)$, so the sequence is in $C_2(k,p)$.

Finally we show that $|C_2(k,p)| = C_k$. For $1 \leq j \leq k$, put $c_{k,j} = |\{\,(c_1, c_2, \ldots, c_{k-1}) \in C_2(k,p) : c_1 = p - k + j\,\}|$ (which as observed above does not depend on p). We then have

$$|C_2(k,p)| = \sum_{j=1}^{k} c_{k,j}$$

and the recurrence

$$c_{k,j} = \sum_{i=1}^{\min\{j,k-1\}} c_{k-1,i},$$

where $c_{2,1} = c_{2,2} = 1$. Observe that the recurrence implies that $c_{k,k-1} = c_{k,k} = |C_2(k-1,p)|$.

Put

$$d_{k,j} = \binom{k+j-3}{j-1} - \sum_{i=0}^{j-3} \binom{k+i-1}{i},$$

with $d_{k,1} = 1$. We claim that $c_{k,j} = d_{k,j}$ for all $k \geq j$; we prove this by induction on j. Clearly $c_{k,1} = 1$ and $c_{k,2} = k-1$. Suppose that $j > 2$ and $c_{k,j-1} = d_{k,j-1}$ for all k. For $k \geq j$ we then have $c_{k+1,j} - c_{k,j} = c_{k+1,j-1} = d_{k+1,j-1}$ and

$$d_{k+1,j} - d_{k,j} = \binom{k+j-3}{j-2} - \sum_{i=1}^{j-3} \binom{k+i-1}{i-1} = d_{k+1,j-1}.$$

Also, $d_{j,j} - c_{j,j} = d_{j,j} - c_{j,j-1} = d_{j,j} - d_{j,j-1} = \binom{2j-3}{j-1} - \binom{2j-4}{j-2} - \binom{2j-4}{j-3} = \binom{2j-3}{j-1} - \binom{2j-4}{j-2} - \binom{2j-4}{j-1} = 0$. Thus, by induction on k, $c_{k,j} = d_{k,j}$ for the given j and all k, and by induction on j this holds for all j.

It now remains only to show that $d_{k,k-1} = C_{k-1}$ for all k. For this, observe that $C_{k-1}/\binom{2k-4}{k-2} = \binom{2k-2}{k-1}/k\binom{2k-4}{k-2} = 2(2k-3)/k(k-1)$. We have

$$d_{k,k-1} = \binom{2k-4}{k-2} - \sum_{i=0}^{k-4} \binom{k+i-1}{i}$$

and

$$\sum_{i=0}^{k-4} \binom{k+i-1}{i} = \binom{2k-4}{k-4}$$

so that $d_{k,k-1}/\binom{2k-4}{k-2} = 1 - \binom{2k-4}{k-4}/\binom{2k-4}{k-2} = 1 - (k-2)(k-3)/k(k-1) = 2(2k-3)/k(k-1) = C_{k-1}/\binom{2k-4}{k-2}$. Thus $d_{k,k-1} = C_{k-1}$. □

We now describe the conjectured extrema with given ℓ_i. We define sets S_j of integers: put $S_0 = \{\, i : 0 \le i \le k-2 \,\}$; put $S_k = \{\, i : n-k+1 \le i \le n-1 \,\}$; and for $1 \le j \le k-1$, put $S_j = \{\, i : s_j - k + 1 \le i \le s_j + k - 2 \,\}$. Put $S = \cup_{j=0}^{k} S_j$. Then S is the union of the sets of the $k-1$ values (or positions) at either end of each of the subsequences in the canonical extremum.

Write the canonical extremum as $d_0, d_1, \ldots, d_{n-1}$. We describe an extremum $c_0, c_1, \ldots, c_{n-1}$. For $i \notin S$, we have $c_i = d_i$; observe (as would be expected, given the symmetries of the problem) that $[0, n-1] \setminus S = \{\, d_i : i \notin S \,\}$.

For $i \le i \le k-1$, let A_i and B_i be arbitrary elements of $C(k, s_i + k - 2)$; let A'_i be $S_i \setminus A_i$ in descending order, and let B'_i be $S_i \setminus B_i$ in descending order. Given this choice of A_i and B_i (there being C_k^{2k-2} possible such choices), we can now describe the extremum associated with the A_i and B_i.

We will define sets L_i for $1 \le i \le k$ and R_i for $0 \le i \le k-1$. Put $L_1 = S_0$ and $R_{k-1} = S_k$. For $1 \le i \le k-1$, put $R_{i-1} = A_i$ and $L_{i+1} = A'_i$. Now, the values of c_i for $i \in S_0$ are the values of R_0 in descending order; the values of c_i for $i \in S_k$ are the values of L_k in descending order; the values of c_i for $i \in B_j$ are the values of L_j in descending order; and the values of c_i for $i \in B'_j$ are the values of R_j in descending order. Observe that this sequence can be divided into k disjoint monotone descending subsequences, of the required lengths; the i^{th} of them, for $1 \le i \le k$, contains R_{i-1}, the fixed values c_j for $s_{i-1} + k - 1 \le j \le s_i - k$, and L_i. Call this subsequence T_i.

An example extremum with $n = 17$ and $k = 3$ is shown in Table 9.2,

Table 9.2: Structure of an example extremal permutation

n	17
k	3
Extremum	5 4 2 12 1 0 9 8 7 16 6 3 15 14 13 11 10
ℓ_1, ℓ_2, ℓ_3	5, 6, 6
s_0, s_1, s_2, s_3	0, 5, 11, 17
Canonical extremum	4 3 2 1 0 10 9 8 7 6 5 16 15 14 13 12 11
Fixed and variable values	X X 2 X X X X 8 7 X X X X 14 13 X X
S_0, S_1, S_2, S_3	$\{0,1\}$, $\{3,4,5,6\}$, $\{9,10,11,12\}$, $\{15,16\}$
S	$\{0,1,3,4,5,6,9,10,11,12,15,16\}$
A_1, A_2	$\{5,4\}$, $\{12,9\}$
B_1, B_2	$\{5,4\}$, $\{11,10\}$
A'_1, A'_2	$\{6,3\}$, $\{11,10\}$
B'_1, B'_2	$\{6,3\}$, $\{12,9\}$
L_1, L_2, L_3	$\{1,0\}$, $\{6,3\}$, $\{11,10\}$
R_0, R_1, R_2	$\{5,4\}$, $\{12,9\}$, $\{16,15\}$
T_1, T_2, T_3	5 4 2 1 0, 12 9 8 7 6 3, 16 15 14 13 11 10

along with the various parameters for its structure described above.

It remains to prove that this sequence has the expected number of monotone subsequences of length $k+1$, and that all extrema (subject to the sequence being divisible into k disjoint monotone descending subsequences) have that form. The description of the sequence makes sense for $n \geq k(2k-2)$, and Theorem 9.7 applies for all such n, but if $n < k(2k-1)$ there can be other extrema not of the form described.

Theorem 9.7 *The sequences just described have exactly $M_k(n)$ monotone subsequences of length $k+1$, all of them descending.*

Proof By the division into k disjoint monotone descending subsequences, of lengths ℓ_i, there are no monotone ascending subsequences of length $k+1$, and there are at least $M_k(n)$ monotone descending subsequences of length $k+1$ (that is, those subsequences entirely within any one of the k subsequences into which the sequence is divided). Thus it is only necessary to prove that there is no monotone descending subsequence of length $k+1$ containing values from more than one of the k subsequences.

If $j \geq i+2$, then the whole of T_j is to the right of the whole of T_i, and all the values in T_j are greater than all the values in T_i. Thus any additional monotone subsequence of length $k+1$ can contain values from only two of the T_j, say T_i and T_{i+1}. If it contains c_p from T_i and c_q from T_{i+1}, we still have $p < q$ except possibly for c_p from L_i and c_q from R_i, and $c_p < c_q$ except possibly for c_p from R_{i-1} and c_q from L_{i+1}. Thus this sequence contains no values from the fixed central regions of T_i and T_{i+1}; if it contains a value from R_{i-1}, then it contains a value from L_{i+1}, and all values are from R_{i-1} and L_{i+1}; if it contains a value from L_i, then all values are from L_i and R_i. But a monotone descending subsequence of length $k+1$ in

R_{i-1} followed by L_{i+1} would be such a subsequence in A_i followed by A'_i, contradicting the definition of $C(k,p)$. Likewise, a monotone descending sequence (of values, as the position goes up) in L_i and R_i may be seen to be equivalent to a monotone descending sequence of positions, as the value goes up, in the positions (going down) of L_i followed by those of R_i; that is, in B_i followed by B'_i, again a contradiction. Thus there are no such monotone subsequences. □

Theorem 9.8 *For $n \geq k(2k-1)$, the sequences which contain no monotone ascending $(k+1)$-subsequences and a minimum number of monotone descending $(k+1)$-subsequences are exactly the $\binom{k}{n \bmod k} C_k^{2k-2}$ sequences described above. The sequences which contain no monotone descending $(k+1)$-subsequences and a minimum number of monotone ascending $(k+1)$-subsequences are those sequences, reversed.*

Proof The derivation of extremal sequences with only ascending $(k+1)$-subsequences from those with only descending $(k+1)$-subsequences is clear. As observed above, sequences with only descending $(k+1)$-subsequences are just those divisible into at most k disjoint monotone descending subsequences, and minimality requires that there be exactly k such subsequences, and that their lengths by $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$. Thus the sequences described above are extremal (from Theorem 9.7), and it is only necessary to show that there are no more extremal sequences.

Suppose $c_0, c_1, \ldots, c_{n-1}$ is an extremal sequence. Suppose that one of the k monotone descending subsequences into which it is divided occupies positions $a_0 < a_1 < \cdots < a_{\ell_i-1}$ (so has values $c_{a_0} > c_{a_1} > \cdots > c_{a_{\ell_i-1}}$), and another occupies positions $b_0 < b_1 < \cdots < b_{\ell_j-1}$, where $a_0 < b_0$. Then $c_{a_0} < c_{b_0}$ (since otherwise $c_{a_0}, c_{b_0}, c_{b_1}, \ldots, c_{b_{k-1}}$ would be another monotone descending $(k+1)$-subsequence), so $c_{a_m} \le c_{a_0} < c_{b_0}$ for all m. Thus $b_0 > a_{\ell_i-k}$, since otherwise c_{b_0}, $c_{a_{\ell_i-k}}$, $c_{a_{\ell_i-k+1}}$, $\ldots$, $c_{a_{\ell_i-1}}$ would be a monotone descending $(k+1)$-subsequence; and $a_{\ell_i-1} > b_{k-1}$, since otherwise either c_{b_0}, c_{b_1}, $\ldots$, $c_{b_{k-1}}$, $c_{a_{\ell_i-1}}$ or c_{a_0}, c_{a_1}, $\ldots$, $c_{a_{k-1}}$, $c_{b_{k-1}}$ would be a monotone descending $(k+1)$-subsequence (depending on the order of $c_{a_{\ell_i-1}}$ and $c_{b_{k-1}}$).

Thus, if we order our k subsequences by the position of the first element, we have seen that the only possible overlap in positions is between the last $k-1$ of one sequence and the first $k-1$ of a later sequence. Because $n \ge k(2k-1)$, each sequence has $\ell_i - 2(k-1) > 0$ central elements that are not in the first or last $k-1$; so the ordering by where the first elements are is the same as the ordering by where the central elements are (which was chosen previously as the ordering of the ℓ_i). In particular, we see that the only overlap in positions is between the last $k-1$ of one sequence and the first $k-1$ of the very next sequence in this order.

Likewise, we may consider the possible overlap in values. If as above we have $i < j$, a_p the positions of sequence i and b_q the positions of sequence j, then suppose for some p, q we have $c_{a_p} > c_{b_q}$. If $p \ge k-1$, then c_{a_0}, c_{a_1}, $\ldots$, $c_{a_{k-1}}$, c_{b_q} would be monotone descending; if $q \le \ell_j - k$, then c_{a_p}, $c_{b_{\ell_j-k}}$, $c_{b_{\ell_j-k+1}}$, $\ldots$, $c_{b_{\ell_j-1}}$ would be monotone descending. Thus the only possible overlap in values is between the first $k-1$ of one sequence and the last $k-1$ of a later sequence, which again must be the very next sequence.

Given these restrictions on overlap of positions, the i^{th} sequence must include the positions from $s_{i-1} + k - 1$ to $s_i - k$ (with $k-1$ positions to either side). The restrictions on overlap of values imply that in these central $\ell_i - 2(k-1)$ positions there must be the canonical values d_i. Thus all extrema have those fixed values that were fixed in our description of the extrema.

For $1 \le i \le k$, let R_{i-1} be the set of the first $k-1$ values in the i^{th} sequence, and let L_i be the set of the last $k-1$ values. Then the i^{th} sequence contains the values R_{i-1}, the fixed values c_j for $s_{i-1}+k-1 \le j \le s_i - k$, and L_i, as in the above description of extrema. Further, the restriction on the overlap of values implies that $L_1 = S_0$ and $R_{k-1} = S_k$, and that, for $1 \le i \le k-1$, R_{i-1} and L_{i+1} are disjoint subsets of $[s_i - k + 1, s_i + k - 2]$. Put $A_i = R_{i-1}$ and $A'_i = L_{i+1}$. Similarly, the positions in our sequence of the values in L_i and R_i are disjoint subsets of $[s_i - k + 1, s_i + k - 2]$; let B_i be the set of positions of the values in L_i, and let B'_i be the set of positions of the values in R_i.

If A_i and B_i are indeed elements of $C(k, s_i + k - 2)$, then the sequence is of the given form, with those A_i and B_i. However, if A_i is not an element of $C(k, s_i + k - 2)$, then the sequence of the values of $A_i = R_{i-1}$ in descending order, followed by those of $A'_i = L_{i+1}$ in descending order, has a monotone descending subsequence of length $k+1$, which is such a subsequence in our original sequence, contradicting minimality. Likewise, if B_i is not an element of $C(k, s_i + k - 2)$, then the sequence of the values of B_i in descending order (the positions of L_i, in ascending order of value), followed by those of B'_i in descending order (the positions of R_i, in ascending order of value), has a monotone descending $(k+1)$-subsequence; that is, there is a monotone descending $(k+1)$-sequence of positions, the values in which are increasing, which gives a monotone descending sequence of values in the original sequence. □

If $n < k(2k-1)$, the above proof no longer works, since some of the k subsequences have no fixed middle elements. However, for $k(2k-2) \le n < k(2k-1)$, the construction still gives sequences with $M_k(n)$ monotone $(k+1)$-subsequences—but there can be other extrema (in which all monotone $(k+1)$-subsequences go in the same direction) as well.

Table 9.3: Number of extremal permutations for $3 \leq k \leq 4$

	$k = 3$		$k = 4$	
n	Total	Both	Total	Both
1	1	0	1	0
2	2	0	2	0
3	6	0	6	0
4	22	0	24	0
5	86	0	118	0
6	306	0	668	0
7	882	0	4124	0
8	1764	0	26328	0
9	1764	0	165636	0
10	8738	0	985032	0
11	6892	0	5323032	0
12	1682	0	25038288	0
13	14706	10092	97173648	0
14	4182	0	288576288	0
15	1250	0	577152576	0
16	6250	2500	577152576	0
17	3750	0	2855608848	0
18	1250	0	2330017568	0
19			710429200	0

Computation shows that, for some n and k, such other extrema do indeed exist. In particular, this applies for $k = 3$ and $12 \leq n < 15$: for each such n there are extrema, in which all monotone $(k + 1)$-subsequences go in the same direction, that are not of the form described above. Further, if we remove the constraint that all monotone $(k+1)$-subsequences go in the same direction, the extremal function is as conjectured for $k = 3$ and $n \leq 18$, and for $k = 4$ and $n \leq 19$ (that is, there are no sequences with fewer than $M_k(n)$ monotone $(k + 1)$-subsequences). For $k = 3$ and $15 \leq n \leq 18$, the extrema described above are found, but when $n = 16$ there are some additional extrema which contain both ascending and descending monotone $(k + 1)$-subsequences. (The first such extremum lexicographically is '4 3 9 2 1 0 13 8 7 6 5 15 14 12 11 10'.) Table 9.3 shows the number of extrema found in each case, in the columns headed 'Total', and the number of those which contain both ascending and descending monotone $(k + 1)$-subsequences, in the columns headed 'Both'.

For larger n exhaustive search could not be done, but heuristic computation, taking a random permutation and attempting to move from that to an apparent extremum, did not find any other cases of apparent extrema (i.e., permutations with $M_k(n)$ monotone subsequences of length $k + 1$) not matching the form described above, nor any sequences with fewer than $M_k(n)$ monotone $(k + 1)$-subsequences, for $n \geq k(2k - 1)$.

The method for the heuristic computation started with a random permutation. Various operations were then applied to it: transposing a pair of values in the permutation; reversing the order of a block of values in the permutation; rotating a block of values (in consecutive positions) in the permutation left or right; and the dual operation of rotating a block of positions (of consecutive values). All possible operations that reduced the number of monotone $(k+1)$-subsequences were considered, if there were any; if there were none, operations that kept the number of monotone $(k+1)$-subsequences the same were considered; in that case, a completely random move was occasionally chosen instead (to try to avoid the problem of being stuck at a local minimum that was not a global minimum). This process was stopped when the permutation had no more than $M_k(n)$ monotone $(k+1)$-subsequences. In computations for various n and k with $n \geq k(2k-1)$, no cases were found with fewer than $M_k(n)$ monotone $(k+1)$-subsequences, and the only extrema found in which not all monotone $(k+1)$-subsequences went in the same direction were with $k = 3$ and $n = 16$. These computations were done for $k = 3$ and $15 \leq n \leq 30$, and for $k = 4$ and $28 \leq n \leq 40$.

Appendices

Appendix A

Tables and source code for Lemma 5.6

A.1 Tables

In Chapter 5, we presented a proof of an inequality (Lemma 5.6) that relied on numerical computation of bounds in a large number of cases. Here we present tables of the bounds computed. The source code of the program that generated these tables is in Section A.2.

Table A.1: Bounds on $\frac{\partial^2 f}{\partial \alpha^2}$ for $0 \leq q \leq 0.4$

$q_{\min}$	$q_{\max}$	$\alpha_{\min}$	$\alpha_{\max}$	$\partial^2_{\min}$	$\partial^2_{\max}$
0	0.4	0	$\frac{1}{4}$	−4.000000	−0.298494
0	0.4	$\frac{1}{4}$	$\frac{3}{8}$	−4.000000	−0.529831
0	0.4	$\frac{3}{8}$	$\frac{7}{16}$	−4.000000	−0.390413
0	0.4	$\frac{7}{16}$	$\frac{1}{2}$	−4.000000	−0.101764

Table A.2: Bounds for $0.4 \leq q \leq 0.48$

$q_{\min}$	$q_{\max}$	$\alpha_{\min}$	$\alpha_{\max}$	$\partial^2_{\min}$	$\partial^2_{\max}$	$\partial_{\min}(\alpha_{\min})$	$\partial_{\max}(\alpha_{\min})$	$f_{\min}(\alpha_{\max})$	$f_{\max}(\alpha_{\max})$
0.4	0.48	$\frac{0}{32}$	$\frac{1}{32}$	−1.507705	−0.868127	0.169366	0.633552	0.004445	0.019798
0.4	0.48	$\frac{1}{32}$	$\frac{2}{32}$	−1.460135	−0.816579	0.142237	0.586436	0.008092	0.038125
0.4	0.48	$\frac{2}{32}$	$\frac{3}{32}$	−1.409836	−0.765012	0.116719	0.540807	0.010993	0.055025
0.4	0.48	$\frac{3}{32}$	$\frac{4}{32}$	−1.365339	−0.714791	0.092813	0.496749	0.013195	0.070548
0.4	0.48	$\frac{4}{32}$	$\frac{5}{32}$	−1.327749	−0.668062	0.070475	0.454083	0.014745	0.084738
0.4	0.48	$\frac{5}{32}$	$\frac{6}{32}$	−1.291636	−0.628018	0.049599	0.412590	0.015682	0.097632
0.4	0.48	$\frac{6}{32}$	$\frac{7}{32}$	−1.258632	−0.601366	0.029973	0.372227	0.016031	0.109264

0.4	0.48	$\frac{7}{32}$	$\frac{8}{32}$	−1.230934	−0.590225	0.011180	0.332895	0.015804	0.119667
0.4	0.48	$\frac{8}{32}$	$\frac{9}{32}$	−1.211104	−0.580139	−0.007264	0.294428	0.015011	0.128868
0.4	0.48	$\frac{9}{32}$	$\frac{10}{32}$	−1.198827	−0.535069	−0.025394	0.256581	0.013695	0.136886
0.4	0.48	$\frac{10}{32}$	$\frac{11}{32}$	−1.182100	−0.415480	−0.042115	0.219118	0.011973	0.143733
0.4	0.48	$\frac{11}{32}$	$\frac{12}{32}$	−1.140395	−0.193886	−0.055098	0.182177	0.010062	0.149426
0.4	0.48	$\frac{12}{32}$	$\frac{13}{32}$	−1.182572	0.098926	−0.061157	0.146540	0.008150	0.154006
0.4	0.48	$\frac{13}{32}$	$\frac{14}{32}$	−1.186989	0.410351	−0.058066	0.109584	0.006336	0.157430
0.4	0.48	$\frac{14}{32}$	$\frac{15}{32}$	−1.168614	0.657096	−0.045242	0.072491	0.004922	0.159696
0.4	0.48	$\frac{15}{32}$	$\frac{16}{32}$	−1.151091	0.790659	−0.024708	0.035972	0.004150	0.160820

Table A.3: Bounds for $0.48 \leq q \leq 0.5$

$q_{\min}$	$q_{\max}$	$\alpha_{\min}$	$\alpha_{\max}$	$\partial^2_{\min}$	$\partial^2_{\max}$	$\partial_{\min}(\alpha_{\min})$	$\partial_{\max}(\alpha_{\min})$
0.48	0.5	$\frac{0}{64}$	$\frac{1}{64}$	−1.408230	−1.213716	0.239120	0.403462
0.48	0.5	$\frac{1}{64}$	$\frac{2}{64}$	−1.384527	−1.187189	0.220156	0.381458
0.48	0.5	$\frac{2}{64}$	$\frac{3}{64}$	−1.360114	−1.160157	0.201606	0.359825
0.48	0.5	$\frac{3}{64}$	$\frac{4}{64}$	−1.335026	−1.132696	0.183479	0.338573
0.48	0.5	$\frac{4}{64}$	$\frac{5}{64}$	−1.309306	−1.104899	0.165780	0.317714

0.48	0.5	$\frac{5}{64}$	$\frac{6}{64}$	−1.283016	−1.076882	0.148516	0.297256
0.48	0.5	$\frac{6}{64}$	$\frac{7}{64}$	−1.256231	−1.048782	0.131690	0.277209
0.48	0.5	$\frac{7}{64}$	$\frac{8}{64}$	−1.229047	−1.020766	0.115303	0.257580
0.48	0.5	$\frac{8}{64}$	$\frac{9}{64}$	−1.201584	−0.993031	0.099353	0.238376
0.48	0.5	$\frac{9}{64}$	$\frac{10}{64}$	−1.173989	−0.965464	0.083837	0.219601
0.48	0.5	$\frac{10}{64}$	$\frac{11}{64}$	−1.147909	−0.937643	0.068752	0.201258
0.48	0.5	$\frac{11}{64}$	$\frac{12}{64}$	−1.123369	−0.909815	0.054101	0.183322
0.48	0.5	$\frac{12}{64}$	$\frac{13}{64}$	−1.099521	−0.882967	0.039885	0.165769
0.48	0.5	$\frac{13}{64}$	$\frac{14}{64}$	−1.076647	−0.857503	0.026089	0.148589
0.48	0.5	$\frac{14}{64}$	$\frac{15}{64}$	−1.054954	−0.832877	0.012690	0.131766
0.48	0.5	$\frac{15}{64}$	$\frac{16}{64}$	−1.034245	−0.807094	−0.000323	0.115283
0.48	0.5	$\frac{16}{64}$	$\frac{17}{64}$	−1.013303	−0.776203	−0.012934	0.099123
0.48	0.5	$\frac{17}{64}$	$\frac{18}{64}$	−0.989288	−0.734603	−0.025062	0.083290
0.48	0.5	$\frac{18}{64}$	$\frac{19}{64}$	−0.957649	−0.676263	−0.036540	0.067832
0.48	0.5	$\frac{19}{64}$	$\frac{20}{64}$	−0.912850	−0.596326	−0.047107	0.052869
0.48	0.5	$\frac{20}{64}$	$\frac{21}{64}$	−0.849667	−0.492464	−0.056425	0.038606
0.48	0.5	$\frac{21}{64}$	$\frac{22}{64}$	−0.764558	−0.365542	−0.064119	0.025330
0.48	0.5	$\frac{22}{64}$	$\frac{23}{64}$	−0.656591	−0.219517	−0.069831	0.013383

0.48	0.5	$\frac{23}{64}$	$\frac{24}{64}$	−0.527732	−0.060788	−0.073261	0.003124
0.48	0.5	$\frac{24}{64}$	$\frac{25}{64}$	−0.382543	0.102733	−0.074211	−0.005122
0.48	0.5	$\frac{25}{64}$	$\frac{26}{64}$	−0.227493	0.266095	−0.072606	−0.011099
0.48	0.5	$\frac{26}{64}$	$\frac{27}{64}$	−0.088946	0.429870	−0.068448	−0.014653
0.48	0.5	$\frac{27}{64}$	$\frac{28}{64}$	0.035108	0.585497	−0.061731	−0.016043
0.48	0.5	$\frac{28}{64}$	$\frac{29}{64}$	0.143961	0.718313	−0.052583	−0.015495
0.48	0.5	$\frac{29}{64}$	$\frac{30}{64}$	0.231379	0.822463	−0.041359	−0.013245
0.48	0.5	$\frac{30}{64}$	$\frac{31}{64}$	0.292463	0.894044	−0.028508	−0.009630
0.48	0.5	$\frac{31}{64}$	$\frac{32}{64}$	0.323857	0.930475	−0.014539	−0.005060

Table A.4: Bounds for $0.5 \leq q \leq 0.55$

$q_{\min}$	$q_{\max}$	$\alpha_{\min}$	$\alpha_{\max}$	$\partial^2_{\min}$	$\partial^2_{\max}$	$\partial_{\min}(\alpha_{\min})$	$\partial_{\max}(\alpha_{\min})$
0.5	0.55	$\frac{0}{64}$	$\frac{1}{64}$	−1.578014	−1.199428	0.121452	0.428178
0.5	0.55	$\frac{1}{64}$	$\frac{2}{64}$	−1.557062	−1.172509	0.102711	0.403521
0.5	0.55	$\frac{2}{64}$	$\frac{3}{64}$	−1.535210	−1.144971	0.084390	0.379192
0.5	0.55	$\frac{3}{64}$	$\frac{4}{64}$	−1.512451	−1.116871	0.066500	0.355205
0.5	0.55	$\frac{4}{64}$	$\frac{5}{64}$	−1.488781	−1.088285	0.049049	0.331573

0.5	0.55	$\frac{5}{64}$	$\frac{6}{64}$	−1.464206	−1.059305	0.032045	0.308311
0.5	0.55	$\frac{6}{64}$	$\frac{7}{64}$	−1.438742	−1.030045	0.015493	0.285432
0.5	0.55	$\frac{7}{64}$	$\frac{8}{64}$	−1.412418	−1.000644	−0.000601	0.262952
0.5	0.55	$\frac{8}{64}$	$\frac{9}{64}$	−1.385274	−0.971269	−0.016237	0.240883
0.5	0.55	$\frac{9}{64}$	$\frac{10}{64}$	−1.357373	−0.942121	−0.031413	0.219238
0.5	0.55	$\frac{10}{64}$	$\frac{11}{64}$	−1.328795	−0.913393	−0.046133	0.198029
0.5	0.55	$\frac{11}{64}$	$\frac{12}{64}$	−1.299646	−0.884563	−0.060405	0.177267
0.5	0.55	$\frac{12}{64}$	$\frac{13}{64}$	−1.270059	−0.855292	−0.074226	0.156960
0.5	0.55	$\frac{13}{64}$	$\frac{14}{64}$	−1.245036	−0.824604	−0.087590	0.137115
0.5	0.55	$\frac{14}{64}$	$\frac{15}{64}$	−1.220577	−0.789540	−0.100475	0.117661
0.5	0.55	$\frac{15}{64}$	$\frac{16}{64}$	−1.195318	−0.744631	−0.112811	0.098590
0.5	0.55	$\frac{16}{64}$	$\frac{17}{64}$	−1.167012	−0.683183	−0.124446	0.079913
0.5	0.55	$\frac{17}{64}$	$\frac{18}{64}$	−1.131434	−0.597688	−0.135121	0.061678
0.5	0.55	$\frac{18}{64}$	$\frac{19}{64}$	−1.082693	−0.482643	−0.144460	0.044000
0.5	0.55	$\frac{19}{64}$	$\frac{20}{64}$	−1.014446	−0.336355	−0.152001	0.027083
0.5	0.55	$\frac{20}{64}$	$\frac{21}{64}$	−0.921518	−0.161436	−0.157257	0.011232
0.5	0.55	$\frac{21}{64}$	$\frac{22}{64}$	−0.801255	0.035759	−0.159779	−0.003167
0.5	0.55	$\frac{22}{64}$	$\frac{23}{64}$	−0.654199	0.246316	−0.159220	−0.015686

0.5	0.55	$\frac{23}{64}$	$\frac{24}{64}$	−0.484007	0.460107	−0.155372	−0.025908
0.5	0.55	$\frac{24}{64}$	$\frac{25}{64}$	−0.296809	0.666970	−0.148182	−0.033471
0.5	0.55	$\frac{25}{64}$	$\frac{26}{64}$	−0.100312	0.857566	−0.137761	−0.038109
0.5	0.55	$\frac{26}{64}$	$\frac{27}{64}$	0.097117	1.026778	−0.124362	−0.039676
0.5	0.55	$\frac{27}{64}$	$\frac{28}{64}$	0.287213	1.177975	−0.108318	−0.038158
0.5	0.55	$\frac{28}{64}$	$\frac{29}{64}$	0.420332	1.308571	−0.089912	−0.033671
0.5	0.55	$\frac{29}{64}$	$\frac{30}{64}$	0.520205	1.416793	−0.069466	−0.027103
0.5	0.55	$\frac{30}{64}$	$\frac{31}{64}$	0.589474	1.494816	−0.047328	−0.018975
0.5	0.55	$\frac{31}{64}$	$\frac{32}{64}$	0.624918	1.534205	−0.023972	−0.009764

A.2 Source code

The following is the source code of the program that generated the tables in Section A.1 and so verifies the computational part of the proof of Lemma 5.6. This source code is in the ISO C language [25].

```
/* Verify inequality associated with quasi-random graphs
   and complete minors.  */
/* Copyright 2002 Joseph Samuel Myers.
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   DATA, OR PROFITS; OR BUSINESS INTERRUPTION) HOWEVER CAUSED
   AND ON ANY THEORY OF LIABILITY, WHETHER IN CONTRACT,
   STRICT LIABILITY, OR TORT (INCLUDING NEGLIGENCE OR
   OTHERWISE) ARISING IN ANY WAY OUT OF THE USE OF THIS
   SOFTWARE, EVEN IF ADVISED OF THE POSSIBILITY OF SUCH DAMAGE.
*/
```

```
#include <math.h>
#include <stdarg.h>
#include <stdio.h>
#include <stdlib.h>

static long double q0; /* q_0 = 0.543689012692...,
                          real root of
                          q^3 + q^2 + q - 1 = 0.  */
static long double p0; /* p_0 = 1 - q_0,
                          real root of
                          p^3 - 4p^2 + 6p - 2 = 0.  */
static long double em1; /* e^-1.  */
static long double em2; /* e^-2.  */
static long double em2x4; /* 4e^-2.  */

typedef struct {
  long double min;
  long double max;
} BOUNDS;

/* Print an error and exit.  */
static void
die(const char *format, ...)
{
  va_list args;
  fprintf(stderr, "minors-quasi-ineq: ");
  va_start(args, format);
  vfprintf(stderr, format, args);
  va_end(args);
  exit(EXIT_FAILURE);
}
```

```
/* Return -r log r.  */
static long double
mrlogr (long double r)
{
  if (r < 0.0L || r > 1.0L)
    die("mrlogr out of range");
  if (r == 0.0L)
    return 0.0L;
  return -r * logl(r);
}

/* Return r (log r)^2.  */
static long double
rlogr2 (long double r)
{
  long double l;
  if (r < 0.0L || r > 1.0L)
    die("rlogr2 out of range");
  if (r == 0.0L)
    return 0.0L;
  l = logl(r);
  return r * l * l;
}

/* Return x^(1/y^2).  */
static long double
powov2 (long double x, long double y)
{
  if (x < 0.0L || x >= 1.0L || y < 0.0L || y > 1.0L)
    die("powov2 out of range");
  if (y == 0.0L || x == 0.0L)
    return 0.0L;
  return powl(x, 1.0L / y / y);
}
```

```
/* Return the bounds of -r log r.  */
static BOUNDS
mlog_bounds (BOUNDS r)
{
  BOUNDS ret;
  long double left;
  long double right;
  left = mrlogr(r.min);
  right = mrlogr(r.max);
  ret.min = (left < right ? left : right);
  if (r.min <= em1 && em1 <= r.max)
    ret.max = em1;
  else
    ret.max = (left > right ? left : right);
  return ret;
}

/* Return the bounds of r (log r)^2.  */
static BOUNDS
log2_bounds (BOUNDS r)
{
  BOUNDS ret;
  long double left;
  long double right;
  left = rlogr2(r.min);
  right = rlogr2(r.max);
  ret.min = (left < right ? left : right);
  if (r.min <= em2 && em2 <= r.max)
    ret.max = em2x4;
  else
    ret.max = (left > right ? left : right);
  return ret;
}
```

```
/* Return the bounds of d^2f/d\alpha^2.  */
static BOUNDS
d2f_bounds (BOUNDS qb, BOUNDS ab)
{
  BOUNDS qoa; /* Bounds of q^(1/\alpha^2).  */
  BOUNDS qoal1; /* ... * - log self.  */
  BOUNDS qoal2; /* ... * log^2 self.  */
  BOUNDS qo1a; /* Bounds of q^(1/(1-\alpha)^2).  */
  BOUNDS qo1al1; /* ... * - log self.  */
  BOUNDS qo1al2; /* ... * log^2 self.  */
  BOUNDS ret;
  qoa.min = powov2(qb.min, ab.min);
  qoa.max = powov2(qb.max, ab.max);
  qo1a.min = powov2(qb.min, 1.0L - ab.max);
  qo1a.max = powov2(qb.max, 1.0L - ab.min);
  qoal1 = mlog_bounds(qoa);
  qoal2 = log2_bounds(qoa);
  qo1al1 = mlog_bounds(qo1a);
  qo1al2 = log2_bounds(qo1a);
  ret.min = (-4.0L
             + 2.0L * qoa.min + 2.0L * qoal1.min
             + 4.0L * qoal2.min
             + 2.0L * qo1a.min + 2.0L * qo1al1.min
             + 4.0L * qo1al2.min);
  ret.max = (-4.0L
             + 2.0L * qoa.max + 2.0L * qoal1.max
             + 4.0L * qoal2.max
             + 2.0L * qo1a.max + 2.0L * qo1al1.max
             + 4.0L * qo1al2.max);
  return ret;
}
```

```
/* Print a fraction in reduced form.
   The only factors to remove are powers of 2.  */
static void
print_reduced_fraction(int n, int d)
{
  while (n % 2 == 0 && d % 2 == 0) {
    n /= 2;
    d /= 2;
  }
  if (n == 0)
    printf("$0$");
  else
    printf("$\\frac{%d}{%d}$", n, d);
}

/* Attempt to prove second derivative always negative
   for given q.  */
static void
prove_d2f_neg (BOUNDS qb)
{
  long double amin, astep;
  amin = 0.0L;
  astep = 0.5L;
  while (amin < 0.5L) {
    BOUNDS ab;
    BOUNDS tb;
    ab.min = amin;
    ab.max = amin + astep;
    tb = d2f_bounds(qb, ab);
    if (tb.max < 0.0L) {
      printf("$%Lg$ & $%Lg$ & ",
             qb.min, qb.max);
      print_reduced_fraction(ab.min / astep, 1.0L / astep);
      printf(" & ");
      print_reduced_fraction(ab.max / astep, 1.0L / astep);
      printf(" & $%.6Lf$ & $%.6Lf$ \\\\\n",
             tb.min, tb.max);
      amin += astep;
    } else {
```

```
      astep /= 2;
      if (astep < 1.0L / 4096.0L)
        break;
    }
  }
  if (amin >= 0.5L) {
    printf("Succeeded in proving second derivative "
           "negative for all alpha,\n"
           "%Lf <= q <= %Lf\n", qb.min, qb.max);
  } else {
    die("FAILED to prove second derivative "
        "negative for all alpha,\n"
        "%Lf <= q <= %Lf", qb.min, qb.max);
  }
}


/* Attempt to prove f always positive for given q.  */
static void
prove_f_pos (BOUNDS qb)
{
  /* Divide [ 0, 1/2 ] into some number of parts.  Bound
     second derivative on each part.  We know first
     derivative is zero at centre; bound it on each part.
     We know f is zero at alpha = 0; bound it at end of
     each part.  Want: f > 0 in centre.  Use this from say
     q = 0.4 up, but not too far up (too near q0).  */
  int parts = 2;
  BOUNDS d2b[32];
  BOUNDS d1b[32]; /* 1st derivative on a region.  */
  BOUNDS d1bleft[32]; /* 1st derivative to
                         the left of a region.  */
  BOUNDS fb[32], fbright[32];
  while (parts <= 32) {
    int i;
    long double d = 0.5L / (long double)parts;
    BOUNDS rb, lb;
    for (i = 0; i < parts; i++) {
      BOUNDS ab;
      ab.min = d * i;
      ab.max = d * (i + 1);
```

```
      d2b[i] = d2f_bounds(qb, ab);
    }
    rb.min = 0.0L;
    rb.max = 0.0L;
    for (i = parts - 1; i >= 0; i--) {
      d1bleft[i].min = rb.min - d * d2b[i].max;
      d1bleft[i].max = rb.max - d * d2b[i].min;
      d1b[i].min = (rb.min < d1bleft[i].min
                    ? rb.min
                    : d1bleft[i].min);
      d1b[i].max = (rb.max > d1bleft[i].max
                    ? rb.max
                    : d1bleft[i].max);
      rb = d1bleft[i];
    }
    lb.min = 0.0L;
    lb.max = 0.0L;
    for (i = 0; i < parts; i++) {
      fbright[i].min = lb.min + d * d1b[i].min;
      fbright[i].max = lb.max + d * d1b[i].max;
      fb[i].min = (lb.min < fbright[i].min
                   ? lb.min
                   : fbright[i].min);
      fb[i].max = (lb.max > fbright[i].max
                   ? lb.max
                   : fbright[i].max);
      if (fbright[i].min <= 0.0L)
        break;
      lb = fbright[i];
    }
    if (i == parts)
      break;
    parts *= 2;
  }
  if (parts <= 32) {
    int i;
    for (i = 0; i < parts; i++) {
      printf("$%Lg$ & $%Lg$ "
             "& $\\frac{%d}{%d}$ & $\\frac{%d}{%d}$ "
             "& $%.6Lf$ & $%.6Lf$ & $%.6Lf$ & $%.6Lf$ "
             "& $%.6Lf$ & $%.6Lf$ \\\\\n",
```

```
             qb.min, qb.max, i, 2 * parts, i + 1, 2 * parts,
             d2b[i].min, d2b[i].max,
             d1bleft[i].min, d1bleft[i].max,
             fbright[i].min, fbright[i].max);
    }
    printf("Proved f always positive for "
           "%Lf <= q <= %Lf, %d steps.\n",
           qb.min, qb.max, parts);
  } else {
    die("FAILED to prove f always positive for "
        "%Lf <= q <= %Lf.",
        qb.min, qb.max);
  }
}

/* Attempt to prove f of a certain shape for given q.  */
static void
prove_f_shape (BOUNDS qb)
{
  /* Divide [ 0, 1/2 ] into some number of parts.  Bound
     second derivative on each part.  We know first
     derivative is zero at centre; bound it on each part.
     Want: first a region with second derivative negative,
     then the first derivative negative until the centre
     (zero at the centre - so then allow a region with
     second derivative positive).  */
  int parts = 2;
  BOUNDS d2b[32];
  BOUNDS d1b[32]; /* 1st derivative on a region.  */
  BOUNDS d1bleft[32]; /* 1st derivative to
                         the left of a region.  */
  while (parts <= 32) {
    int i;
    long double d = 0.5L / (long double)parts;
    BOUNDS rb;
    for (i = 0; i < parts; i++) {
      BOUNDS ab;
      ab.min = d * i;
      ab.max = d * (i + 1);
      d2b[i] = d2f_bounds(qb, ab);
    }
```

```
    rb.min = 0.0L;
    rb.max = 0.0L;
    for (i = parts - 1; i >= 0; i--) {
      d1bleft[i].min = rb.min - d * d2b[i].max;
      d1bleft[i].max = rb.max - d * d2b[i].min;
      d1b[i].min = (rb.min < d1bleft[i].min
                    ? rb.min
                    : d1bleft[i].min);
      d1b[i].max = (rb.max > d1bleft[i].max
                    ? rb.max
                    : d1bleft[i].max);
      rb = d1bleft[i];
    }
    for (i = 0; i < parts; i++) {
      if (d2b[i].max >= 0)
        break;
    }
    for (; i < parts; i++) {
      if (d1b[i].max >= 0)
        break;
    }
    for (; i < parts; i++) {
      if (d2b[i].min <= 0)
        break;
    }
    if (i == parts)
      break;
    parts *= 2;
  }
  if (parts <= 32) {
    int i;
    for (i = 0; i < parts; i++) {
      printf("$%Lg$ & $%Lg$ "
             "& $\\frac{%d}{%d}$ & $\\frac{%d}{%d}$ "
             "& $%.6Lf$ & $%.6Lf$ & $%.6Lf$ & $%.6Lf$ \\\\\n",
             qb.min, qb.max, i, 2 * parts, i + 1, 2 * parts,
             d2b[i].min, d2b[i].max,
             d1bleft[i].min, d1bleft[i].max);
    }
    printf("Proved f correct shape for "
           "%Lf <= q <= %Lf, %d steps.\n",
```

```
          qb.min, qb.max, parts);
  } else {
    die("FAILED to prove f correct shape for "
        "%Lf <= q <= %Lf.",
        qb.min, qb.max);
  }
}


int
main (void)
{
  /* Compute q0 by cubic formula.  */
  q0 = (cbrtl(3.0L * sqrtl(33.0L) + 17.0L)
        - cbrtl(3.0L * sqrtl(33.0L) - 17.0L)
        - 1.0L) / 3.0L;
  p0 = 1.0L - q0;
  em1 = expl(-1.0L);
  em2 = expl(-2.0L);
  em2x4 = 4 * em2;
  prove_d2f_neg((BOUNDS) { 0.0L, 0.4L });
  prove_f_pos((BOUNDS) { 0.4L, 0.48L });
  prove_f_shape((BOUNDS) { 0.48L, 0.5L });
  prove_f_shape((BOUNDS) { 0.5L, 0.55L });
  exit(EXIT_SUCCESS);
}
```

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