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Graphs without Large Complete Minors are Quasi-Random

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We answer a question of Sós by showing that, if a graph G of order n and density p has no complete minor larger than would be found in a random graph $G(n,p)$, then G is quasi-random, provided either $p > 0.45631\ldots$ or $\kappa(G) \geqslant n(\log\log\log n)/(\log\log n)$, where $0.45631\ldots$ is an explicit constant.

The results proved can also be used to fill the gaps in an argument of Thomason, describing the extremal graphs having no K_t minor for given t.

1. Introduction

As usual, define a graph H to be a *minor* of a graph G (writing $H \prec G$) if H can be obtained from G by a series of vertex and edge deletions and edge contractions; or, equivalently, if there are disjoint subsets $W_u \subseteq V(G)$, for $u \in V(H)$, such that all $G[W_u]$ are connected and, for all $uv \in E(H)$, there is an edge in G between W_u and W_v.

Fernandez de la Vega [4] noticed from Bollobás, Catlin and Erdős [1] (see below) that random graphs are good examples of graphs with high average degree but no large complete minor. Kostochka [5, 6] showed that they are within a constant factor of being optimal. More recently, Thomason [12] essentially determined the extremal function for complete minors K_t in terms of the average degree, as $t \to \infty$: if we define

$$c(t) = \min\{c \,:\, e(G) \geqslant c|G| \text{ implies } K_t \prec G\},$$

then $c(t)$ exists, and he showed that $c(t) = (\alpha + o(1))t\sqrt{\log t}$, where $\alpha = 0.3190863\ldots$ is an explicit constant; or, equivalently, that the minimum average degree guaranteeing a K_t minor is $(2\alpha + o(1))t\sqrt{\log t}$.

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Bollobás, Catlin and Erdős [1] showed that the largest K_t minor in a random graph $G(n,p)$ has

$$t = (1+o(1))\frac{n}{\sqrt{\log_{1/q} n}},$$

where $q = 1-p$. Choosing $q = \lambda = 0.2846681\ldots$, another explicit constant, and $n = t\sqrt{\log_{1/\lambda} t}$, gives examples of graphs with average degree $(2\alpha + o(1))t\sqrt{\log t}$ and no K_t minor. Examples with the same average degree and larger order are then constructed by taking many disjoint copies of $G(n, 1-\lambda)$.

Thomason's proof in [12] therefore consists of showing that a graph (not necessarily random) of average degree greater than $(2\alpha + o(1))t\sqrt{\log t}$ must have a K_t minor. Having proved this, he then claimed at the end of the paper, with an outline proof, that any extremal graph (that is, a graph with average degree $(2\alpha + o(1))t\sqrt{\log t}$ and no K_t minor) is essentially the example given above: that (save for a few edges) it consists of a disjoint union of quasi-random graphs of the order and density given above. Here 'quasi-random' is used in the sense of Chung, Graham and Wilson [3] or Thomason [10]: that is, that every induced subgraph of order $|G|/2$ (or more generally $\alpha|G|$ for any constant α) has essentially the same density.

Sós asked a more general question about complete minors and quasi-randomness. It is sometimes the case that quasi-random graphs contain larger minors than the corresponding random graphs; examples are given by Thomason [11], and indeed the problem, raised by Mader, of explicitly presenting graphs without large complete minors remains open. Sós asked whether, however, the converse might be true: that if a graph of order n and density p had no complete minor larger than that in a random graph $G(n,p)$, would the graph then necessarily be quasi-random?

At first sight, the outline argument in Section 7 of [12] would appear to be usable to address Sós's question. The relevant part of the argument is, essentially, that if G is of maximal density having no K_t minor, then no subgraph of order $(1-\epsilon)|G|$ can have density much greater than that of G, or it would have a larger minor than that found in the whole of G. Thus G is quasi-random. This argument is, however, flawed on two counts: first, if the argument is quantified properly, using the method and results of [10], it turns out that the minor in the subgraph is not as large as is required; and second, the argument does not rule out the possibility of graphs G with very sparse subgraphs, and there are non-quasi-random graphs (such as some bipartite graphs) that have no large subgraph with significantly larger density than the original graph, but do have a few large subgraphs with significantly smaller density.

In this paper, our purpose is to answer Sós's question; and at the same time, our results provide enough information to fill in the gaps in Thomason's argument.

The answer to Sós's question turns out to depend on the density and connectivity of G. A graph G of order n and density p that is not quasi-random will have a complete minor larger than that of a random graph $G(n,p)$ if p is large (including $p \geqslant \frac{1}{2}$), and the same result holds for smaller p provided that G has moderate connectivity. Otherwise, if both the density and the connectivity are small, the assertion may fail; for example, the disjoint union of two $G(n/2, \frac{1}{2})$ random graphs has order n and density $\frac{1}{4}$ but does not have a complete minor as large as that of $G(n, \frac{1}{4})$.

Throughout this paper, we shall generally follow the notation of Bollobás [2]; the following additional notation will also be useful. Given a graph G whose vertex set is partitioned into two disjoint subsets X, Y, we define the three densities

$$p_X = \frac{e(X)}{\binom{|X|}{2}}, \qquad p_{XY} = \frac{e(X,Y)}{|X||Y|}, \qquad p_Y = \frac{e(Y)}{\binom{|Y|}{2}},$$

where $e(X)$, $e(Y)$ and $e(X,Y)$ are, respectively, the number of edges of G spanned by X, spanned by Y and joining X to Y. We likewise put $q_X = 1 - p_X$, $q_{XY} = 1 - p_{XY}$ and $q_Y = 1 - p_Y$. It is the principal feature of quasi-random graphs that, for every X of given order, the value of p_X differs little from p, the density of G itself, which of course implies that all of p_X, p_{XY} and p_Y are close to p.

A precise statement of the answer to Sós's question can now be given. This involves a constant $p_0 = \frac{1}{3}\big(4 + \sqrt[3]{3\sqrt{33} - 17} - \sqrt[3]{3\sqrt{33} + 17}\big) = 0.45631\ldots$, which is the real root of $x^3 - 4x^2 + 6x - 2 = 0$; and $q_0 = 1 - p_0$ is the real root of $x^3 + x^2 + x - 1 = 0$. (This arises from the inequality $q^4 - 2q + 1 = (q-1)(q^3 + q^2 + q - 1) > 0$; as long as this inequality holds, a random graph on half the vertices with twice the density will have a larger minor than a random graph on all the vertices, but when $q > q_0$ such a random graph on half the vertices will have a smaller minor, and the extremal graphs become the graphs made up of multiple disjoint random graphs with a few extra edges, described above, rather than being themselves random graphs.)

Theorem 1.1. *Given $\epsilon > 0$, there exist $\delta > 0$ and N with the following property.*

Let G be a graph of order $n > N$ and edge density p, where $\epsilon < p < 1 - \epsilon$. Suppose that G has a vertex partition (X, Y) with $|X| = |Y|$ such that at least one of $|p_X - p|$, $|p_{XY} - p|$ and $|p_Y - p|$ exceeds ϵ. Suppose that either

$$p > p_0 + \epsilon \tag{1.1}$$

or

$$\kappa(G) \geqslant n(\log\log\log n)/(\log\log n). \tag{1.2}$$

Then G contains a K_t minor for

$$t > (1+\delta)\frac{n}{\sqrt{\log_{1/q} n}},$$

where, as usual, $q = 1 - p$.

Roughly, this states that a non-quasi-random graph has a minor larger than a corresponding random graph provided that one of the conditions (1.1) or (1.2) holds.

In fact, provided we consider only graphs of reasonably connectivity (1.2), we can make a much more precise statement about the minimum order of a complete minor.

Let G be a graph of order n with a vertex partition (X, Y), where $|X| = \alpha|G|$. Let q_X, q_{XY}, q_Y be as above. Let $p = 1 - q$ be the density of G. Then, if n is large, we have essentially

$$q = \alpha^2 q_X + (1-\alpha)^2 q_Y + 2\alpha(1-\alpha) q_{XY}.$$

Consider now a constrained random graph G' of order n with a fixed vertex partition (X, Y), where the edges are chosen independently and at random, with probabilities p_X inside X, p_{XY} between X and Y and p_Y inside Y. It is straightforward to adapt the arguments of Bollobás, Catlin and Erdős [1] to show that the maximum order of a complete minor in this constrained random graph is

$$(1+o(1))\frac{n}{\sqrt{\log_{1/q_*} n}},$$

where

$$q_* = {q_X}^{\alpha^2} {q_Y}^{(1-\alpha)^2} {q_{XY}}^{2\alpha(1-\alpha)}.$$

Taking logarithms and applying Jensen's inequality, we see that

$$q \geqslant q_*,$$

with equality if and only if $q_X = q_Y = q_{XY}$.

The following theorem shows that our graph G with its given partition will have a complete minor at least as large as found in the corresponding constrained random graph G', provided that G has reasonable connectivity.

Theorem 1.2. *Let $0 < \epsilon < 1$. Then there exists N with the following property.*

Let G be a graph of order $n > N$, with vertex partition (X, Y) as above, $|X| = \alpha n$, where $\epsilon < \alpha < 1-\epsilon$. Let q_X, q_Y, q_{XY} and q_* be defined as above, and suppose $\epsilon < q_X, q_Y, q_{XY} \leqslant 1$ and $q_* < 1-\epsilon$. Suppose $\kappa(G) \geqslant n(\log\log\log n)/(\log\log n)$. Then $G \succ K_s$, where

$$s = \left\lceil (1-\epsilon)\frac{n}{\sqrt{\log_{1/q_*} n}} \right\rceil.$$

This theorem is an extension of Theorem 4.1 of Thomason [12], which gives

$$s \geqslant (1-\epsilon)\frac{n}{\sqrt{\log_{1/q} n}},$$

when G has density p and reasonable connectivity; that theorem follows from Theorem 1.2 because $q \geqslant q_*$. The same inequality also means that Theorem 1.2 implies Theorem 1.1 for graphs of reasonable connectivity, except for extreme values of the parameters.

2. Outline of proof

We prove Theorem 1.2 first; then from it we derive Theorem 1.1. To prove Theorem 1.2, we must partition $V(G)$ into s parts $W_1, \ldots, W_s$, such that each $G[W_i]$ is connected and there is an edge in G between each W_i and W_j. The critical aspect is finding a partition that ensures that there are edges between each pair of parts of the minor; if such edges exist, the parts can be made connected, provided that G itself is reasonably connected.

For the case considered in Thomason [12], where all that is known about G is its density p (and that G is reasonably connected, where appropriate), that paper gives an argument for constructing a partition with the desired properties. The principal feature is to order the vertices of G by degree and to use this ordering to take a suitably constrained random partition.

At first sight it would appear that, to extend this argument to the present case, where the existing partition (X, Y) and the densities p_X, p_Y and p_{XY} must be taken into account, would require a two-dimensional partial ordering of vertices by degrees to both X and Y; but such an argument is not strong enough to yield the required results. Nevertheless, somewhat surprisingly, it turns out that the argument can be adapted to the present case after all; although ordering the vertices by degree is not appropriate, there is a suitable function on the vertices which provides a single linear order that will work. Having found this ordering, the argument then follows somewhat similar lines to those of Thomason's proof of Theorem 4.1 in [12].

Having proved Theorem 1.2, Theorem 1.1 is derived as follows: either G is reasonably connected, in which case the result is immediate, or G has a very small cutset (and we require $q < q_0$ to go any further). If this cutset splits the graph into reasonably sized parts (each with at least $\frac{1}{50}$ of the vertices), we show that (for $q < q_0$) one of these parts is sufficiently much denser than the original graph that it would be expected to have a larger minor than a random graph of the same order and density as the original graph. If small cutsets only cut small numbers of vertices off the graph, we remove vertices of small degree; either only a few of them exist, so after removing them the resulting graph cannot have small parts cut off by small cutsets, or many exist, and after removing enough of them the resulting graph has a larger density. We iterate this process a bounded number of times, if necessary, ending up at a graph of large connectivity and with a large complete minor, and so deduce Theorem 1.1 using Theorem 1.2.

3. Proof of Theorem 1.2

We define a *complete equipartition* of G to be a partition of $V(G)$ into disjoint parts $W_1, \dots, W_k$, such that G contains an edge from W_i to W_j for all $1 \leqslant i < j \leqslant k$ and such that $\lfloor |G|/k \rfloor \leqslant |W_i| \leqslant \lceil |G|/k \rceil$ for all i. The following lemma lies at the heart of the paper.

Lemma 3.1. *Let G be a graph of order n with α, X, Y, q, q_X, q_Y, q_{XY}, q_* as above. Let ℓ, $s \geqslant 2$ be integers with $n = s\ell$ and $\ell\alpha$ an integer, $\alpha\ell \geqslant 2$, $(1-\alpha)\ell \geqslant 2$. Then G contains a complete equipartition into at least*

$$s - \frac{4s}{\omega\eta} - 2s^2(18\omega)^\ell \left[\frac{q_*}{1-\eta}\right]^{(1-\eta)\ell/(\ell - \max(1/\alpha, 1/(1-\alpha)))}$$

parts, for every $0 < \eta \leqslant 1 - {q_X}^\alpha {q_{XY}}^{(1-\alpha)}, 1 - {q_Y}^{(1-\alpha)} {q_{XY}}^\alpha$ and $\omega \geqslant 1$.

Proof. For a vertex $v \in V(G)$ we define $Q(v; X) = \{ x \in X - \{v\} : vx \notin E(G) \}$, the set of non-neighbours of v (other than v itself) within X, and $Q(v; Y) = \{ y \in Y - \{v\} : vy \notin E(G) \}$, the set of non-neighbours of v (other than v itself) in Y Also put $Q(v) =$

$Q(v;X) \cup Q(v;Y)$. For $W \subset V(G)$, put $N(W) = \{ u \in V(G) : W \subset Q(u) \}$. Let

$$q(v;X) = |Q(v;X)|/(\alpha n - 1),$$
$$q(v;Y) = |Q(v;Y)|/((1-\alpha)n - 1).$$

Put

$$r(v) = q(v;X)^{\alpha\ell} q(v;Y)^{(1-\alpha)\ell}.$$

Then order the vertices of X as $x_1, \ldots, x_{\alpha n}$ in order of increasing $r(x_i)$, and similarly order the vertices of Y as $y_1, \ldots, y_{(1-\alpha)n}$ in order of increasing $r(y_i)$.

Now define blocks $B_j^X = \{ x_i : (j-1)s < i \leqslant js \}$ for $1 \leqslant j \leqslant \alpha\ell$, and $B_j^Y = \{ y_i : (j-1)s < i \leqslant js \}$ for $1 \leqslant j \leqslant (1-\alpha)\ell$. Independently and uniformly choose random permutations β_j^X, β_j^Y of the blocks, and so induce a random partition of $V(G)$ into s parts $W_t = \{ x_{\beta_j^X(t)} : 1 \leqslant j \leqslant \alpha\ell \} \cup \{ y_{\beta_j^Y(t)} : 1 \leqslant j \leqslant (1-\alpha)\ell \}$, $1 \leqslant t \leqslant s$.

Let $S^X \subset X$, $S^Y \subset Y$, $S = S^X \cup S^Y$. Then, for W one of the random parts,

$$\begin{aligned}
\Pr(W \subset S) &= \prod_{j=1}^{\alpha\ell} \frac{|S^X \cap B_j^X|}{s} \prod_{j=1}^{(1-\alpha)\ell} \frac{|S^Y \cap B_j^Y|}{s} \\
&\leqslant \left(\frac{1}{\alpha\ell} \sum_{j=1}^{\alpha\ell} \frac{|S^X \cap B_j^X|}{s} \right)^{\alpha\ell} \left(\frac{1}{(1-\alpha)\ell} \sum_{j=1}^{(1-\alpha)\ell} \frac{|S^Y \cap B_j^Y|}{s} \right)^{(1-\alpha)\ell} \\
&= \left(\frac{|S^X|}{\alpha n} \right)^{\alpha\ell} \left(\frac{|S^Y|}{(1-\alpha)n} \right)^{(1-\alpha)\ell},
\end{aligned}$$

using the AM/GM inequality.

For $S = Q(x_i)$, we have $\Pr(x_i \in N(W)) = \Pr(W \subset S) \leqslant q(x_i;X)^{\alpha\ell} q(x_i;Y)^{(1-\alpha)\ell} = r(x_i)$. Similarly, $\Pr(y_i \in N(W)) \leqslant r(y_i)$. By the ordering of vertices chosen,

$$\mathbf{E}(|B_j^X \cap N(W)|) \leqslant sr(x_{js}),$$

and

$$\mathbf{E}(|B_j^Y \cap N(W)|) \leqslant sr(y_{js}).$$

Say that W *rejects* a block B_j^X (respectively B_j^Y) if $|B_j^X \cap N(W)| > \omega sr(x_{js})$ (respectively $|B_j^Y \cap N(W)| > \omega sr(y_{js})$), so that W rejects a given block with probability at most $1/\omega$; put $R^X(W) = \{ j < \alpha\ell : W \text{ rejects } B_j^X \}$ and $R^Y(W) = \{ j < (1-\alpha)\ell : W \text{ rejects } B_j^Y \}$, so $\mathbf{E}(|R^X(W)|) \leqslant (\alpha\ell - 1)/\omega$ and $\mathbf{E}(|R^Y(W)|) \leqslant ((1-\alpha)\ell - 1)/\omega$. Call a random part W *acceptable* if $|R^X(W)| < \eta(\alpha\ell - 1)$ and $|R^Y(W)| < \eta((1-\alpha)\ell - 1)$, so $\Pr(W \text{ is not acceptable}) < 2/\omega\eta$.

Now let W be some acceptable part; put $M^X(W) = \{1, \ldots, \alpha\ell - 1\} - R^X(W)$, $M^Y(W) = \{1, \ldots, (1-\alpha)\ell - 1\} - R^Y(W)$, $m^X = |M^X(W)| \geqslant (1-\eta)(\alpha\ell - 1)$ and $m^Y = |M^Y(W)| \geqslant (1-\eta)((1-\alpha)\ell - 1)$. Let W' be another random part and let P_W be the probability, conditional on W, of there being no edge from W' to W. Then we have

$$\begin{aligned} P_W &= \Pr(W' \subset N(W) \mid W) \\ &\leqslant \prod_{j \in M^X(W)} \frac{\omega s r(x_{js})}{s-1} \prod_{j \in M^Y(W)} \frac{\omega s r(y_{js})}{s-1} \\ &< (2\omega)^\ell \prod_{j \in M^X(W)} r(x_{js}) \prod_{j \in M^Y(W)} r(y_{js}). \end{aligned}$$

Now, we have

$$\begin{aligned} \left[\prod_{j \in M^X(W)} r(x_{js})^{1/\ell} \right]^{1/m^X} &\leqslant \frac{1}{m^X} \sum_{j \in M^X(W)} r(x_{js})^{1/\ell} \\ &= \frac{1}{m^X} \sum_{j \in M^X(W)} q(x_{js}; X)^\alpha q(x_{js}; Y)^{(1-\alpha)} \\ &\leqslant \frac{1}{m^X s} \sum_{i=1}^{\alpha n} q(x_i; X)^\alpha q(x_i; Y)^{(1-\alpha)} \\ &\leqslant \frac{1}{m^X s} \left[\sum_{i=1}^{\alpha n} q(x_i; X) \right]^\alpha \left[\sum_{i=1}^{\alpha n} q(x_i; Y) \right]^{(1-\alpha)} \\ &= \frac{\alpha n {q_X}^\alpha {q_{XY}}^{(1-\alpha)}}{m^X s} \\ &\leqslant \frac{{q_X}^\alpha {q_{XY}}^{(1-\alpha)}}{1-\eta} . \frac{\alpha \ell}{\alpha \ell - 1} \end{aligned}$$

(using Hölder's inequality) and likewise

$$\left[\prod_{j \in M^Y(W)} r(y_{js})^{1/\ell} \right]^{1/m^Y} \leqslant \frac{{q_Y}^{(1-\alpha)} {q_{XY}}^\alpha}{1-\eta} . \frac{(1-\alpha)\ell}{(1-\alpha)\ell - 1},$$

whence

$$\begin{aligned} P_W &\leqslant (2\omega)^\ell \left[\frac{{q_X}^\alpha {q_{XY}}^{(1-\alpha)}}{1-\eta} . \frac{\alpha \ell}{\alpha \ell - 1} \right]^{\ell m^X} \left[\frac{{q_Y}^{(1-\alpha)} {q_{XY}}^\alpha}{1-\eta} . \frac{(1-\alpha)\ell}{(1-\alpha)\ell - 1} \right]^{\ell m^Y} \\ &\leqslant (18\omega)^\ell \left[\frac{q_*}{1-\eta} \right]^{(1-\eta)\ell(\ell - \max(1/\alpha, 1/(1-\alpha)))} \\ &= P, \quad \text{say}. \end{aligned}$$

Now, we have a partition with at most $4s/\omega\eta$ unacceptable parts and at most $2s^2P$ defective pairs of acceptable parts with no edge between them. Remove each unacceptable part, and one part from each defective pair. This yields an equipartition of part of the graph into the required number of parts, and the remaining vertices may then be distributed among those parts. □

We now convert this lemma into a more usable form.

Lemma 3.2. *Let $0 < \epsilon < 1$. Then there exists N with the following property.*

Let G be a graph of order $n > N$, with vertex partition (X, Y), $|X| = \beta n$, where $\epsilon < \beta < 1-\epsilon$. Let $\epsilon < q_X, q_Y, q_{XY}$ and $q_* < 1-\epsilon$. Then G has a complete equipartition into at least $(1-\epsilon)n/\sqrt{\log_{1/q_*} n}$ parts.

Proof. Suppose n large (sufficiently large for all the parts of this proof to work). Put $d = \lfloor\sqrt{n}\rfloor$. We apply Lemma 3.1 with $\alpha = \lfloor d\beta\rfloor/d$, $\ell = d\left\lceil(1/d)(1+\epsilon/2)\sqrt{\log_{1/q_*} n}\right\rceil$, $s = \lfloor n/\ell\rfloor$, $\eta = \epsilon(1-q_*)/8$ and $\omega = 128/\epsilon^2(1-q_*)$. We lose a few vertices from G in the conversion to integer s and ℓ, but only $O\left(\sqrt{\log_{1/q_*} n}\right) < \epsilon^3 n$ of them, so the effect on the n and q_* used in Lemma 3.1 is insignificant.

We have $s > (1-\epsilon/2)n/\sqrt{\log_{1/q_*} n}$, so it will suffice to show that each of the terms subtracted from s in the statement of Lemma 3.1 is at most $\epsilon s/4$; this holds for the first term by choice of η and ω. For the second, we have $\ell(\ell - \max(1/\alpha, 1/(1-\alpha))) > (1+\epsilon)\log_{1/q_*} n$, and since $\eta < \epsilon/8$ we have $(1-\eta)\ell(\ell - \max(1/\alpha, 1/(1-\alpha))) > (1+3\epsilon/4)\log_{1/q_*} n$. Also, $\log(1/(1-\eta)) = -\log(1-\eta) < 2\eta = \epsilon(1-q_*)/4$ since $\eta < 1/8$; and $1 - q_* < \log(1/q_*)$, so $\log(1/(1-\eta)) < (\epsilon/4)\log(1/q_*)$; thus $\log(q_*/(1-\eta)) < (\epsilon/4 - 1)\log(1/q_*)$. Thus,

$$
\begin{aligned}
2s^2(18\omega)^\ell & \left[\frac{q_*}{1-\eta}\right]^{(1-\eta)\ell(\ell-\max(1/\alpha,1/(1-\alpha)))} \\
&\leqslant s\exp\left[\log n + \ell\log(2304/\epsilon^2(1-q_*)) - (1+3\epsilon/4)(1-\epsilon/4)\log n\right] \\
&\leqslant s\exp\left[2\sqrt{\log_{1/q_*} n}\log(2304/\epsilon^2(1-q_*)) - (\epsilon/4)\log n\right] \\
&\leqslant s\exp\left[2\sqrt{(\log n)/(1-q_*)}\log(2304/\epsilon^2(1-q_*)) - (\epsilon/4)\log n\right] \\
&< \epsilon s/4
\end{aligned}
$$

for large n, given the bounds on q_*. □

We now use this result to find complete minors in dense graphs. We use a number of simple lemmas from Thomason [12]. The following are his Proposition 4.1, Lemma 4.1 and Lemma 4.2 respectively, and proofs may be found in [12].

Lemma 3.3. *Let $X \sim Bi(n,p)$ be a binomially distributed random variable. Let $0 < \epsilon < 1$. Then* $\Pr(|X - np| > \epsilon np) < 2e^{-\epsilon^2 np/4}$.

Lemma 3.4. *Given a bipartite graph with vertex classes A and B, wherein each vertex of A has at least $\gamma|B|$ neighbours in B ($\gamma > 0$), then there exists a set $M \subset B$ such that every vertex in A has a neighbour in M, and $|M| \leqslant \lfloor\log_{1/(1-\gamma)}|A|\rfloor + 1$.*

Lemma 3.5. *Let G be a connected graph and let u, $v \in V(G)$. Then u and v are joined in G by at least $\kappa^2(G)/4|G|$ internally disjoint paths of length at most $2|G|/\kappa(G)$.*

Proof of Theorem 1.2. Assume throughout that n is large. By Lemma 3.5, for any u, $v \in V(G)$, u and v are joined in G by at least $\kappa^2/4n$ internally disjoint paths with length at most

$$h = 2(\log\log n)/(\log\log\log n);$$

let $P_{u,v}$ be the set of such paths.

Let $r = 1/(\log\log\log n)$ and select vertices independently and at random with probability r from $V(G)$, forming a set of vertices C, where $|C| < 2rn$ with probability at least 1/2. Using Lemma 3.3, the probability that a given vertex $v \in G$ of degree $\deg(v)$ has more than $\epsilon \deg(v)/6$ neighbours within C is less than $1/n^2$. For given $u, v \in V(G)$, C contains all the internal vertices of some given path in $P_{u,v}$ with probability at least r^h, independently for each such path; and $r^h > (\log n)^{-1/6}$, so $r^h|P_{u,v}|/2 > n/(\log n)^{1/3}$. Again using Lemma 3.3, we conclude that the probability that fewer than $r^h|P_{u,v}|/2$ paths of $P_{u,v}$ lie entirely within C is less than $1/n^3$; so there is some set C (which we now fix) with $|C| < 2rn$, with every vertex v of G having at most $\epsilon \deg(v)/6$ neighbours inside C, and every pair u, v of vertices of G having at least $n/(\log n)^{1/3}$ internally disjoint paths from u to v, with length at most h, whose internal vertices lie within C.

Similarly, choose a random subset D of $V(G)-C$, choosing each vertex with probability r. With probability at least 1/2 we have $|D| < 2rn$; any given vertex v has at least $\deg(v)/2 \geqslant \kappa/2$ neighbours outside C and the probability that more than $\epsilon \deg(v)/6$ of these or fewer than $r\kappa/4$ of these lie in D is at most $1/n^2$; so we may fix D such that every vertex v has between $r\kappa/4$ and $\epsilon \deg(v)/6$ neighbours in D.

Now consider the graph $G-C-D$, and apply Lemma 3.2 to it with parameter $\epsilon/8$. Each of q_X, q_Y, q_{XY} has changed by at most $\epsilon^2/10$, so we may find a complete equipartition of $G-C-D$ into s parts, say $W_1', \dots, W_s'$. By s applications of Lemma 3.4 we find disjoint subsets $M_1, \dots, M_s$ in D such that every vertex of W_i' has a neighbour in M_i and $|M_i| \leqslant 5(\log\log n)^2$ for all i. We have $s < n(\log\log n)/\sqrt{\log n}$, so after $M_1, \dots, M_j$ have been chosen every vertex of $G-C-D$ has at least $r\kappa/4 - 5s(\log\log n)^2 > r\kappa/8$ neighbours in D; so that the conditions of that lemma apply with $A = W_{j+1}'$, $B = D - M_1 - \cdots - M_j$ and $\gamma = r\kappa/8|D| > 1/(8\log\log n)$; and, since A was a part in an equipartition of $G-C-D$ into s parts, $(1-\epsilon)|A| \leqslant \sqrt{(\log n)/\log(1/q_*)} \leqslant \sqrt{(\log n)/(1-q_*)}$; so we have M_{j+1} with $|M_{j+1}| \leqslant 1 + \log_{1/(1-\gamma)}|A| \leqslant 1 + (\log|A|)/\gamma < 5(\log\log n)^2$.

It now remains to find disjoint $N_1, \dots, N_s$ in C such that $M_i \cup N_i$ is connected (then, $W_i = W_i' \cup M_i \cup N_i$ will give our complete minor). We can find such N_i with $|N_i| \leqslant 5h(\log\log n)^2$, since, given $N_1, \dots, N_j$, we have $|N_1 \cup \cdots N_j| < 5sh(\log\log n)^2$ and we have $n/(\log n)^{1/3}$ paths of length at most h with internal vertices in C between any pair of vertices u, v of M_{j+1}, so we find $|M_{j+1}| - 1$ such paths to connect M_{j+1}. □

4. Proof of Theorem 1.1

From now on, we aim only for minors of order $(1+\delta)n/\sqrt{\log_{1/q} n}$, not for stronger results involving q_*. Theorem 1.2 now yields Theorem 1.1 in the well-connected case.

Lemma 4.1. *Let $\epsilon > 0$ be given. Then there exist $\delta > 0$ and N with the following property. Let G be a graph of order $n > N$ and edge density p, where $\epsilon < p < 1-\epsilon$. Suppose that*

G has a vertex partition (X, Y) with $|X| = |Y|$, such that at least one of $|p_X - p|$, $|p_{XY} - p|$ and $|p_Y - p|$ exceeds ϵ. Suppose that $\kappa(G) \geqslant n(\log\log\log n)/(\log\log n)$. Then G contains a K_t minor for $t > (1+\delta)n/\sqrt{\log_{1/q} n}$ (where, as usual, $q = 1 - p$).

Proof. Since $\log q = \log(\alpha^2 q_X + 2\alpha(1-\alpha)q_{XY} + (1-\alpha)^2 q_Y)$ and $\log q_* = \alpha^2 \log q_X + 2\alpha(1-\alpha)\log q_{XY} + (1-\alpha)^2 \log q_Y$, we can, by considering the graph of $\log x$, choose small ϵ_1 (much smaller than ϵ) and $\delta > 0$ such that, if $q \geqslant \epsilon/2$ and if any of $|q_X - q|$, $|q_Y - q|$, $|q_{XY} - q|$ exceeds $\epsilon/4$, then $(1-\epsilon_1)(\log(1/q_{**}))^{1/2} > (1+\delta)(\log(1/q))^{1/2}$ holds, where we define $q_{**} = \max(\epsilon_1, q_X)^{\alpha^2} \max(\epsilon_1, q_Y)^{(1-\alpha)^2} \max(\epsilon_1, q_{XY})^{2\alpha(1-\alpha)}$.

If, now, $\epsilon_1 < q_X, q_Y, q_{XY}$, this lemma follows by applying Theorem 1.2 to G with ϵ_1 in place of ϵ. If we have one of $q_X, q_Y, q_{XY} \leqslant \epsilon_1$ (but nevertheless $q > \epsilon$), then this means that almost all edges are present in some part of the graph, and q_* is much smaller than q. Remove a few edges from the relevant part or parts of the graph to increase q_X, q_Y, q_{XY} to above ϵ_1; by a result of Mader [7] that a minimal k-connected graph on n vertices ($n \geqslant 3k$) has at most $k(n-k)$ edges, we may easily do this while preserving the required connectivity. Since ϵ_1 is small compared to q, after removing these edges, we still have (in the modified graph) one of $|q_X - q|$, $|q_Y - q|$, $|q_{XY} - q|$ exceeding $\epsilon/4$, so Theorem 1.2 applied to the new graph gives our result. □

It now remains only to consider the case of small connectivity. Define the expected order of a complete minor in a random graph of order n and density of non-edges q to be $t(n, q) = n/\sqrt{\log_{1/q} n}$. In many cases, we will reduce from a graph G of order n and density at least $p = 1 - q$ to a subgraph H of order βn, and want the expected order of a complete minor in H to be as large as that expected in a random graph of order n and edge density at least p; that is, if H is of density $p' = 1 - q'$, we will want $\beta n/\sqrt{\log_{1/q'}(\beta n)} \geqslant n/\sqrt{\log_{1/q} n}$; it will suffice if $\beta\sqrt{\log(1/q')} \geqslant \sqrt{\log(1/q)}$, that is, if $q' \leqslant q^{1/\beta^2}$. Define $q'(q, \beta) = q^{1/\beta^2}$. Similarly, we may want H to have a minor at least $(1+\delta)$ times larger, so we also define $q'(q, \beta, \delta) = q^{(1+\delta)^2/\beta^2}$.

Lemma 4.2. *Let $f_q(\alpha) = 1 - \alpha^2 - (1-\alpha)^2 + \alpha^2 q^{1/\alpha^2} + (1-\alpha)^2 q^{1/(1-\alpha)^2} - q$. If $0 < \alpha < 1$ and $0 \leqslant q < q_0 = 1 - p_0$, then $f_q(\alpha) > 0$. Further, for $0 \leqslant q < q_0$, we have $f_q(\frac{1}{100}) > 10^{-3}$.*

Proof. The behaviour of the function $f_q(\alpha)$ is illustrated by Figure 1, in which graphs of $f_{0.4}$, $f_{0.5}$ and $f_{0.55}$ are shown. A quick glance at this figure makes the lemma appear very plausible. Unfortunately, I do not have a short and elegant proof of the lemma. A full proof exists, but it involves many cases and numerical computation, so is not included here. It may be found in [8] and [9]. □

We now apply this lemma.

Corollary 4.3. *Let $\epsilon > 0$ be given. Then there exist $\delta > 0$ and N with the following property.*

Let G be a graph of order $n > N$ and edge density at least p, where $p_0 + \epsilon < p$. Suppose $\kappa(G) < n(\log\log\log n)/(\log\log n)$, and that there exists a cutset S in G with $|S| = \kappa(G)$

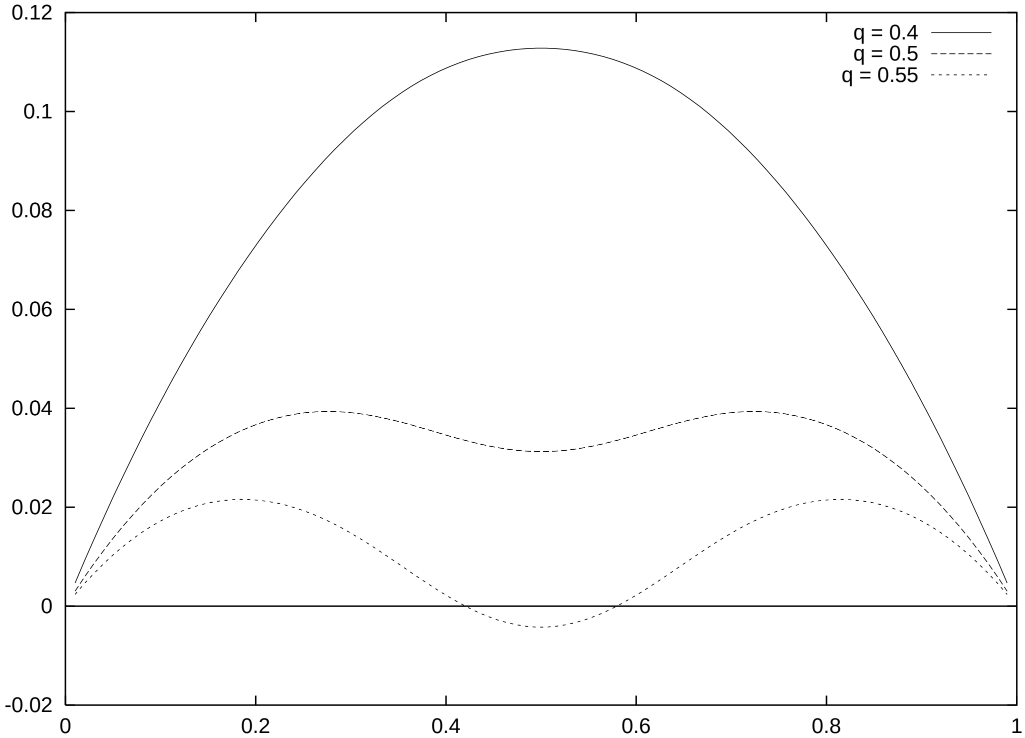

Figure 1 f_q

such that there exist X, Y *with* $V(G) = X \cup Y$, $S = X \cap Y$ *and* $E(G) = E(G[X]) \cup E(G[Y])$, *and* $\frac{1}{100}(n + |S|) \leqslant |X| \leqslant \frac{99}{100}(n + |S|)$. *Then* G *has a subgraph* H *of order at least* $\frac{1}{100}n$ *and at most* $\frac{99}{100}(n + |S|)$ *and density* $p' = 1 - q'$ *where* $q' \leqslant q'(q, |H|/n, \delta)$.

Proof. Suppose we have such a cutset, and let $|S| = \gamma n$. Choose our X, Y. Our subgraph H will be one of $G[X]$ and $G[Y]$. Put $|X| = \alpha(1 + \gamma)n$ and $|Y| = (1 - \alpha)(1 + \gamma)n$, where $\frac{1}{100} \leqslant \alpha \leqslant \frac{99}{100}$.

Accordingly, define p_X, p_Y as the densities of edges in X, Y; so that $p \leqslant \alpha^2(1 + \gamma)^2 p_X + (1 - \alpha)^2(1 + \gamma)^2 p_Y$ and $q \geqslant 1 - \alpha^2(1 + \gamma)^2(1 - q_X) - (1 - \alpha)^2(1 + \gamma)^2(1 - q_Y) = (1 - \alpha^2(1 + \gamma)^2 - (1 - \alpha)^2(1 + \gamma)^2) + \alpha^2(1 + \gamma)^2 q_X + (1 - \alpha)^2(1 + \gamma)^2 q_Y = s$, say.

We want to show that either $q_X \leqslant q'(q, \alpha(1 + \gamma), \delta)$ or $q_Y \leqslant q'(q, (1 - \alpha)(1 + \gamma), \delta)$. Since we have $q \geqslant s$, it will suffice to show that either $q_X \leqslant q'(s, \alpha(1 + \gamma), \delta)$ or $q_Y \leqslant q'(s, (1 - \alpha)(1 + \gamma), \delta)$. Suppose not; we shall derive a contradiction. For, we then have $q_X > q'(s, \alpha(1 + \gamma), \delta)$ and $q_Y > q'(s, (1 - \alpha)(1 + \gamma), \delta)$, so

$$s > (1 - \alpha^2(1 + \gamma)^2 - (1 - \alpha)^2(1 + \gamma)^2) + \alpha^2(1 + \gamma)^2 q'(s, \alpha(1 + \gamma), \delta) + (1 - \alpha)^2(1 + \gamma)^2 q'(s, (1 - \alpha)(1 + \gamma), \delta),$$

that is,

$$\begin{aligned} f(s,\alpha,\gamma,\delta) = (1-\alpha^2(1+\gamma)^2-(1-\alpha)^2(1+\gamma)^2) \\ +\alpha^2(1+\gamma)^2 s^{(1+\delta)^2/\alpha^2(1+\gamma)^2} \\ +(1-\alpha)^2(1+\gamma)^2 s^{(1+\delta)^2/(1-\alpha)^2(1+\gamma)^2} \\ -s \leqslant 0. \end{aligned}$$

This function is continuous in all four variables, and $f(s,\alpha,0,0)$ is $f_s(\alpha)$ in the notation of Lemma 4.2.

By Lemma 4.2, $f_s(\alpha)$ is bounded away from zero on $\frac{1}{100} \leqslant \alpha \leqslant \frac{99}{100}$, $0 \leqslant q \leqslant q_0 - \epsilon$. By continuity (and so uniform continuity), we deduce that we cannot have $f(s,\alpha,\gamma,\delta) \leqslant 0$ for γ, δ sufficiently small (depending on ϵ), so providing our contradiction. □

Corollary 4.4. *Let $\epsilon > 0$ be given. Then there exist $\delta > 0$ and N with the following property.*

Let G be a graph of order $n > N$ and edge density at least p, where $p_0 + \epsilon < p < 1-\epsilon$. Suppose that G has a vertex partition (X', Y') with $|X'| = |Y'|$, such that at least one of $|p_{X'} - p|$, $|p_{X'Y'} - p|$ and $|p_{Y'} - p|$ exceeds ϵ. Suppose that $\delta(G) \geqslant \frac{1}{60}n$. Then either G contains a K_t minor for $t > (1+\delta)n/\sqrt{\log_{1/q} n}$ (where, as usual, $q = 1-p$) or G has a subgraph H of order at least $\frac{1}{100}n$ and at most $\frac{199}{200}n$ and density $p' = 1-q'$ where $q' \leqslant q'(q, |H|/n, \delta)$.

Proof. If $\kappa(G) \geqslant n(\log\log\log n)/(\log\log n)$, we have a large minor by Lemma 4.1. Otherwise, we have a small cutset S, with $|S| = \kappa(G)$, and if we choose any division of G by this cutset, this induces X, Y satisfying the conditions of Corollary 4.3 (since, for any choice of X, Y, where one of X and Y might be too small, some vertex in X has degree at most $|X|$; but the bound on the minimal degrees then implies that $|X|$, $|Y| \geqslant \frac{1}{60}n$). The result then follows by Corollary 4.3. □

Corollary 4.5. *Let $\epsilon > 0$ be given. Then there exists N with the following property.*

Let G be a graph of order $n > N$ and edge density at least p, where $p_0 + \epsilon < p < 1-\epsilon$. Suppose $\delta(G) \geqslant \frac{1}{50}n$. Then either G contains a K_t minor for $t > (1-\epsilon)n/\sqrt{\log_{1/q} n}$ (where, as usual, $q = 1-p$) or G has a subgraph H of order at least $\frac{1}{100}n$ and at most $\frac{199}{200}n$ and density $p' = 1-q'$ where $q' \leqslant q'(q, |H|/n)$.

Proof. If $\kappa(G) \geqslant n(\log\log\log n)/(\log\log n)$, we have a large minor by Theorem 4.1 of [12]. Otherwise, we have a small cutset S, with $|S| = \kappa(G)$, and if we choose any division of G by this cutset, this induces X, Y satisfying the conditions of Corollary 4.3 (since, for any choice of X, Y, where one of X and Y might be too small, some vertex in X has degree at most $|X|$; but the bound on the minimal degrees then implies that $|X|$, $|Y| \geqslant \frac{1}{50}n$). The result then follows by Corollary 4.3. □

We now consider graphs with small minimal degree. For a graph G, let G_ζ be the result of applying the operation 'remove a vertex of minimal degree' $\zeta|G|$ times to G, where each time the vertex removed is of degree less than $\frac{1}{50}n$.

Lemma 4.6. *Let $\epsilon > 0$ be given. Then there exist N and $\delta > 0$ with the following property.*

Let G be a graph of order $n > N$ and edge density at least $p \geqslant p_0$. Suppose $\delta(G) < \frac{1}{50}n$. Let $\zeta \leqslant \frac{1}{50}$, and suppose that G_ζ exists. Then G_ζ has density $p' = 1-q'$ where $q' \leqslant q'(q, 1-\zeta)$. Further, if $\zeta \geqslant \epsilon^2$, then $q' \leqslant q'(q, 1-\zeta, \delta)$.

Proof. We use $\delta = 10^{-3}\epsilon^2$, and, for convenience, put $\delta = 0$ when considering $\zeta < \epsilon^2$.

We have $e(G_\zeta) \geqslant e(G) - \frac{1}{50}\zeta n^2$, so

$$\begin{aligned} p' &\geqslant \left(\frac{1}{2}p - \frac{1}{50}\zeta\right) \Big/ \left(\frac{1}{2}(1-\zeta)^2\right) \\ &\geqslant (1+2\zeta)\left(p - \frac{1}{25}\zeta\right) \\ &\geqslant p + 0.8\zeta \end{aligned}$$

since $p \geqslant p_0$. Thus $q' \leqslant q - 0.8\zeta$.

We want to show that $q' \leqslant q^{(1+\delta)^2/(1-\zeta)^2}$, so it will suffice to show that $(q - 0.8\zeta)^{(1-\zeta)^2} \leqslant q^{(1+\delta)^2}$; that is, $q \times q^{-2\zeta+\zeta^2} \times (1 - 0.8\zeta/q)^{(1-\zeta)^2} \leqslant q \times q^{2\delta+\delta^2}$, or, equivalently, cancelling a factor of q and taking logarithms, that

$$0 > (\log(1/q))(2\zeta - \zeta^2 + 2\delta + \delta^2) + (1 - 2\zeta + \zeta^2)\log(1 - 0.8\zeta/q).$$

We have that $\log(1/q) \leqslant e^{-1}/q < 0.38/q$, and $\log(1 - 0.8\zeta/q) \leqslant -0.8\zeta/q$, so it will suffice to show that

$$\begin{aligned} 0 &> (0.38/q)(2\zeta - \zeta^2 + 2\delta + \delta^2) - (0.8\zeta/q)(1 - 2\zeta + \zeta^2) \\ &= (1/q)(-0.04\zeta + 1.22\zeta^2 - 0.8\zeta^3 + 0.38(2\delta + \delta^2)) \\ &\leqslant (1/q)(-0.03\zeta + 1.22\zeta^2 - 0.8\zeta^3) \end{aligned}$$

by our choice of δ. This result holds provided $\zeta \leqslant 0.025$. □

We now use the above results to show that general graphs of a given density have minors as large as random graphs, if the density is sufficient or a connectivity condition applies.

Lemma 4.7. *Let $\epsilon > 0$ be given. Then there exists N with the following property.*

Let G be a graph of order $n > N$ and edge density at least p, where $0.9999 < p < 1 - \epsilon$. Then G contains a K_t minor for $t > (1-\epsilon)n/\sqrt{\log_{1/q} n}$ (where, as usual, $q = 1 - p$).

Proof. Repeatedly remove the vertex of minimal degree from G, until the minimal degree is at least $\frac{1}{50}n$; say that we have removed ζn vertices. Then $\zeta < \frac{1}{50}$, and G_ζ has density $p' = 1 - q'$ where $q' \leqslant q'(q, 1-\zeta)$ by Lemma 4.6. Put $n' = (1-\zeta)n = |G_\zeta|$.

If $\kappa(G_\zeta) \geqslant n'(\log\log\log n')/(\log\log n')$, then Lemma 4.7 follows from Theorem 4.1 of [12]. So suppose that $\kappa(G_\zeta) < n'(\log\log\log n')/(\log\log n')$. Then, as in the proof of Corollary 4.5, we have a small cutset S, with $|S| = \kappa(G_\zeta)$, and if we choose any division of G_ζ by this cutset, this induces X, Y satisfying the conditions of Corollary 4.3 (since, for any choice of X, Y, where one of X and Y might be too small, some vertex in X

has degree at most $|X|$; but the bound on the minimal degrees then implies that $|X|$, $|Y| \geqslant \frac{1}{50}n'$). However, the density condition on G means that we cannot have such X, Y. □

The next lemma shows that general graphs of a given density have minors as large as random graphs, if the density is sufficient or a connectivity condition applies.

Lemma 4.8. *Let $\epsilon > 0$ be given. Then there exists N with the following property.*

Let G be a graph of order $n > N$ and edge density at least p, where $\epsilon < p < 1 - \epsilon$. Suppose that either $\kappa(G) \geqslant n(\log\log\log n)/(\log\log n)$ or $p > p_0 + \epsilon$. Then G contains a K_t minor for $t > (1-\epsilon)n/\sqrt{\log_{1/q} n}$ (where, as usual, $q = 1 - p$).

Proof. The well-connected case is just Theorem 4.1 of [12]; when $p \geqslant 0.9999$, the result will follow by Lemma 4.7. To prove the general result, we apply a bounded number of operations to our graph, each moving from (H', p') (where initially $(H', p') = (G, p)$) to (H'', p'') where H' is a subgraph of G of density at least p', H'' is a subgraph of G with density at least p'', where $p'' = 1 - q'(1 - p', |H''|/|H'|)$, so ensuring that at all stages $|H'|/\sqrt{\log_{1/q'} |H'|} > n/\sqrt{\log_{1/q} n}$ (where $q' = 1 - p'$). These operations are of the following forms. After any two of these operations have been consecutively applied, the new graph H''' satisfies $\frac{1}{10000}|H'| \leqslant |H'''| \leqslant \frac{199}{200}|H'|$. The upper bound ensures that the number of steps is bounded, because the density must significantly increase; the lower bound ensures that p' stays bounded above by some quantity less than 1, so that Theorem 4.1 of [12] can indeed be applied.

(1) If $p' \geqslant 0.9999$, we have our minor by Lemma 4.7.
(2) If the connectivity is high, $\kappa(H') \geqslant |H'|(\log\log\log|H'|)/(\log\log|H'|)$, we have our minor by Theorem 4.1 of [12].
(3) Otherwise, if some cutset of order $\kappa(H')$ splits the graph into parts each of which has order at least $\frac{1}{60}|H'|$, then the conditions of Corollary 4.3 apply and by Corollary 4.3 we have our (H'', p'') with $|H''| < \frac{199}{200}|H'|$.
(4) Otherwise, $\delta(H') < \frac{1}{50}|H'|$. Remove successively a vertex of minimal degree until all vertices have degree at least $\frac{1}{50}|H'|$ or at least $\frac{1}{50}|H'|$ vertices have been removed, forming the subgraph $H'' = H'_\zeta$. In either case, Lemma 4.6 shows that this subgraph is sufficiently dense. This is the only operation that might not significantly reduce $|H'|$; but if it does not, then $\delta(H'') \geqslant \frac{1}{50}|H''|$, so the next operation must be one of the first three above.

The number of passes through the above loop is bounded, so eventually one of the first two operations listed applies and we have our minor. □

Given this result, we can now prove Theorem 1.1.

Proof of Theorem 1.1. Let $\epsilon > 0$ be small. If $\kappa(G) \geqslant n(\log\log\log n)/(\log\log n)$, the result follows from Lemma 4.1. Otherwise, repeatedly remove a vertex of minimal degree from G, until either the minimal degree is at least $\frac{1}{50}n$ or $\epsilon^2 n$ vertices have been removed.

If $\epsilon^2 n$ vertices have been removed, then by Lemma 4.6 the resulting graph is H', with density at least $p' = 1 - q'$, where $q' = q'(q, 1-\epsilon^2, \delta)$. By Lemma 4.8 applied to H', q' and $\delta/2$, $H' \succ K_t$ where $t \geqslant (1-\delta/2)|H'|/\sqrt{\log_{1/q'}|H'|}$; by the definition of $q'(q, 1-\epsilon^2, \delta)$ we have

$$t \geqslant (1-\delta/2)(1+\delta)n/\sqrt{\log_{1/q} n} > (1+\delta/6)n/\sqrt{\log_{1/q} n},$$

as required.

Otherwise, say ζn vertices were removed, where $\zeta < \epsilon^2$. The numbers removed from X and Y may not be equal, so remove a few more vertices until they are, yielding a subgraph H'; so no more than $2\epsilon^2 n$ vertices are removed in total. For H' of density p', we have that at least one of $|p'_X - p'|$, $|p'_{XY} - p'|$, $|p'_Y - p'|$ exceeds $\epsilon/2$, and $p_0 + \epsilon/2 < p' < 1 - \epsilon/2$. If $\kappa(H') \geqslant |H'|(\log\log\log|H'|)/(\log\log|H'|)$, the result again follows from Lemma 4.1 applied to H' and $\epsilon/2$.

Otherwise, apply Corollary 4.4 to H' and $\epsilon/2$. Either it gives the required minor, or it reduces to a subgraph H'' of density $p'' = 1 - q''$ where $q'' \leqslant q'(q', |H''|/|H'|, \delta)$. Now Lemma 4.8 applied to H'', q'' and $\delta/2$ gives the result, as before. □

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References

[1] Bollobás, B., Catlin, P. A. and Erdős, P. (1980) Hadwiger's conjecture is true for almost every graph. *Europ. J. Combin.* **1** 195–199.

[2] Bollobás, B. (1998) *Modern Graph Theory*, Springer.

[3] Chung, F. R. K., Graham, R. L. and Wilson, R. M. (1989) Quasi-random graphs. *Combinatorica* **9** 345–362.

[4] Fernandez de la Vega, W. (1983) On the maximum density of graphs which have no subcontraction to K^s. *Discrete Math.* **46** 109–110.

[5] Kostochka, A. V. (1982) The minimum Hadwiger number for graphs with a given mean degree of vertices. *Metody Diskret. Analiz.* **38** 37–58. In Russian.

[6] Kostochka, A. V. (1984) Lower bound of the Hadwiger number of graphs by their average degree. *Combinatorica* **4** 307–316.

[7] Mader, W. (1972) Über minimal *n*-fach zusammenhängende, unendliche Graphen und ein Extremalproblem. *Arch. Math.* (*Basel*) **23** 553–560.

[8] Myers, J. S. (2001) An inequality arising in graph minors. `http://www.srcf.ucam.org/~jsm28/publications/2001/minors-inequality.ps`.

[9] Myers, J. S. (2002) Extremal theory of graph minors and directed graphs (provisional dissertation title). PhD dissertation, University of Cambridge. In preparation.

[10] Thomason, A. (1987) Pseudo-random graphs. In *Random Graphs '85* (M. Karoński and Z. Palka, eds), Vol. 33 of *Ann. Discrete Math.*, North-Holland, pp. 307–331.

[11] Thomason, A. (2000) Complete minors in pseudo-random graphs. *Random Struct. Alg.* **17** 26–28.

[12] Thomason, A. (2001) The extremal function for complete minors. *J. Combin. Theory Ser. B* **81** 318–338.