Problems Drive 2001

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1 Problems

Problem 0
In which Linus arranges a fact-finding mission to Britain and finds the Encyclopedia of Integer Sequences wanting.

Determine, with brief explanation, the next four terms of each of the following sequences:

(α) 1, 3, 8, 27, 50, 61, 78, 125

(β) 1, 2, 1, 1, 2, 2, 1, 2, 1, 1, 2, 1

(γ) 2, 3, 8, 27, 125

(δ) 0, 1, 12, 32, 110, 131, 160, 214

(ϵ) 0, 0, 1, -1, 0, 1, 2, 2, 3, 2, 1, 0, 1, 0, -1, 0

Problem 1
In which symbols are shuffled.

In ✆✃✈☞✏✔▲✆✈✃☞✃✏✔▲✔✆✹☞✏ , a triangle represents equality and scissors represent multiplication. The remaining symbols each represent different digits from 1 to 9. What digit does each symbol represent?

Problem 2
In which 29 numbers are named.

Find the correct one-to-one mapping between the elements in these two sets.

(a) A Myriad Myriad

(b) Asankhyeya

(c) Assuming one poppy-head could contain 10000 grains of sand, the number of grains of sand required to fill the universe—according to Archimedes

(d) Brun's Constant

(e) Duotrigintillion

(f) Googol

(g) Googolplex

(h) Graham's Number

(i) Liouville's Number

(j) Mascheroni's Constant

(k) Millimillillion

(l) Number of moves required to transfer the 64 golden disks at the temple at Benares

(m) Pakoti

(n) Plato's Number

(o) Primo-vigesimo-centillion

(p) Probably the number of digits in Archimedes' Number of the Cattle of the Sun

(q) Sexdecillion

(r) Skewes' Number

(s) States of the original Rubik Cube puzzle

(t) Tertio-vigintillion

(u) The Fifth Kaprekar Number

(v) The Number of 6x10 Pentomino Rectangles

(w) The Number of magic squares of order 5 (excluding rotations and reflections)

(x) The Number of the Minor Planet Archimedes

(y) The Page Number of Wiles' Proof of Fermat's Last Theorem, Ann. of Math. (2) **141** (1995), no. 3

(z) The Second Geometric Number of Plato

(aa) The probability of four players at Bridge being dealt a complete suit is 1 in this number

(bb) The smallest Carmichael Number

(cc) Xenocrates' Number of syllables formed from letters of the Greek alphabet

(i) $0.110001000000000000000001000000 0\ldots$

(ii) $0.577215664901532\ldots$

(iii) $1.90195\ldots$

(iv) 216

(v) 297

(vi) 443

(vii) 561

(viii) 2339

(ix) 3600

(x) 206545

(xi) 12960000

(xii) 275305224

(xiii) 1002000000000

(xiv) 18446744073709551615

(xv) 43252003274489856000

(xvi) 2235197406895366368301560000

(xvii) 10^8

(xviii) 10^{14}

(xix) 10^{51}

(xx) 10^{96}

(xxi) 10^{99}

(xxii) 10^{100}

(xxiii) 10^{138}

(xxiv) 10^{140}

(xxv) 10^{366}

(xxvi) $10^{3000003}$

(xxvii) $10^{10^{100}}$

(xxviii) $10^{10^{10^{34}}}$

(xxix) ∞ (well, not quite, but the largest number on this list anyway)

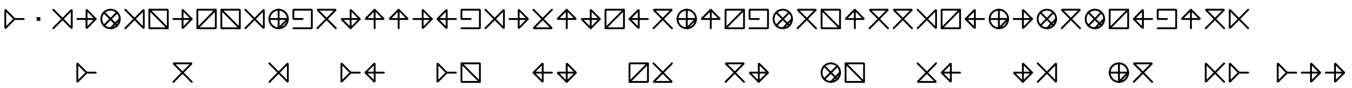
Problem 3
In which bases confuse.

Define $f(x)$ to be the value of the base 2 expansion of x, read as a number in base 3. Compute

$$\int_0^{1/3} f(x)\, dx.$$

Problem 4
In which Z buys an ancient stone.

Z successfully decoded the following inscription by using the second line as a possible hint. State the normal notation for the number represented by the first line.



Problem 5
In which Arthur narrowly avoids a falling rock.

Given a regular icosahedral rock with unit edge length and mass distribution of $1 + d^2$ (where d is the distance from the surface of the icosahedron) extending out over the edge of a horizontal cliff top, what is the maximum surface area of the overhang of the bottom face over the (straight) edge of the cliff such that the slightest breeze could topple it over the edge? (The circumdiameter of the icosahedron with unit edge length is $\sqrt{(5+\sqrt{5})/2}$.)

Problem 6
In which Arthur loses his cheesillionaire status but is reunited with Eugene.

Last year Arthur, my pet mouse, went online to establish a dot com. By July, lastcheese.com had 75025 registered customers. Whenever lastcheese.com sends out an electronic cheeseshot to some of its customers, it costs the business the fixed total sum of E5, regardless of how many customers are being cheesemailed.

Every time that Arthur's long-lost brother Eugene receives a cheesemail, he sends an anonymouse cheesepayment of 250q for a Premium Electronic Cheesesniff. Alas, lastcheese.com still has no other paying customers, and its valuation is falling rapidly.

Arthur wants to find out which cheesemail address on his list belongs to Eugene. So he comes up with a cunning plan. Each day for the next few weeks he will select a subset of the registered customers and only send the cheeseshot to them.

What is the smallest budget (cost of cheeseshots minus any anonymouse cheesepayments received) that Arthur needs in order to guarantee identifying Eugene's cheesemail address uniquely?

[Note for non-residents: The currency of Archimedea is the Eureka, divided into 100 qarches; E1.00=100q.]

Problem 7
In which 2000 Chimeopoleans imagine numbers.

Two months ago, it was so cold that the perfectly circular pond in the hitherto peaceful Archimedean town of Chimeopolis froze over for the first and final time of the century. There is an ancient Chimeopolean tradition that whenever the pond freezes over, one resident for each year since (and including) 1AD must stand, evenly-spaced, around the pond and inscribe a polygon between them on the ice. Then, avoiding collisions, they must each skate across half the diagonals that emanate from the vertices where they are standing, so as to ensure that all the diagonals are marked out on the ice.

The Mayor of the town decided to mark the new Millennium by holding a prize competition. He already knew that the 1000^{th} power of the length of the longest straight line marked out on the ice measured in chimiles was 1000. So he offered a prize of E1000 for the first resident who could tell him the total of the 1000^{th} powers of the lengths of all the lines—edges and diagonals—that were supposed to be on the ice. Within two days the prize was successfully claimed, although it contained factorials which were not multiplied out: what was the answer which clinched the prize?

Problem 8
In which five gross are three gross and nine.

The software in my brand new ZQWTYFXGRNPJ2001000 mobile phone seems to be buggy. Not only does the phone often crash during calls, but its limited 13–digit calculator appears to be incapable of handling multiplication (let alone division!) by single-digit numbers. It seems to be performing such calculations by simply moving the first digit of the number round to the end!

Assuming this is really what it is doing, in what proportion of all the possible multiplications of integers of 13 digits or fewer by a one-digit positive number N that I could ask the equipment to perform would its answer be correct? List all these calculations where N exceeds 1. [No numbers starting with the digit 0 can be entered.]

Problem 9
In which you do it yourself.

Attempting to emulate Archimedes while singing the 1938 Archimedeans' anthem simultaneously, you damage the bathroom floor (presumed infinite) by dropping a heavy crown onto it. After this somewhat expensive accident you wish to retile it with regular polygons of unit edge length using polygons with at least three different numbers of edges. Regular polygonal tiles are available with any number of edges from 3 to 12 inclusive; which of these are **not** suitable for your purpose?

Problem X
In which triangles may be assembled.

During Threes Week all the students at Triplety Hall are issued with three red sticks picked at random from the ones supplied. During manufacture the wooden sticks reach a state where they are all identical. Then a random length of each one is dipped partially into red paint and the red part is snapped off and sold to Triplety Hall.

The students must join the ends of their three sticks together before dropping them into the river which flows under the college and racing to the place where it reemerges to see whose arrives first. Experience shows that the students that form acute-angled triangles usually do best: assuming there is no cheating, what in the expected proportion of students that can fall into this category?

Problem E
In which you guess some scores.

For many years the Archimedeans of Cambridge have held a Problems Drive. This is no leisurely drive through the countryside, but rather an opportunity for the top young mathematicians of each generation to prove to each other the mathematics of which they are made. The competition consists of twelve questions. Each question is marked out of one. Fractional marks are often awarded for partially completed answers. Occasionally infinitessimal bonus marks are awarded.

In the 2001 competition, the final question consisted of two parts. The first part was a multiple choice question containing some statements concerning the scores achieved in the competition. Teams had to determine which of the seven options was the only correct one. Unfortunately the statements are not available.

(a) Only statement 1 is true

(b) Statements 1,2 and 3 are true

(c) Only statements 3 and 4 are true

(d) Exactly two statements are true

(e) At least two statements are false

(f) All the statements are true

(g) All the statements are false

The final question also gave teams an unexpected opportunity to boost their own scores by predicting the scores of:

(a) their own team;

(b) the top-scoring team;

(c) the lowest-scoring team;

in all cases excluding the score on both parts of that question for both calculating the score and determining the teams with the highest and lowest scores.

Draw some unique symbol at the top right hand corner of this sheet, but do not write your names on it.

Answer to Problem 0

(α) Answer:
Explanation:

(β) Answer:
Explanation:

(γ) Answer:
Explanation:

(δ) Answer:
Explanation:

(ϵ) Answer:
Explanation:

Answer to Problem 1

Answer to Problem 2

(a)	(k)	(u)
(b)	(l)	(v)
(c)	(m)	(w)
(d)	(n)	(x)
(e)	(o)	(y)
(f)	(p)	(z)
(g)	(q)	(aa)
(h)	(r)	(bb)
(i)	(s)	(cc)
(j)	(t)	

Answer to Problem 3

Answer to Problem 4

Answer to Problem 5

Answer to Problem 6

Answer to Problem 7

Answer to Problem 8

Answer to Problem 9

Answer to Problem X

Answer to Problem E

Correct option:

Scores:

(a)

(b)

(c)

2 Answers and mark scheme

Answer to Problem 0

Mark scheme: 1/20 for each correct value. Explanations ignored unless they provide a plausible reason behind an answer other than that listed as correct here, in which case the alternative answer is accepted.

(α) Answer: 144, 167, 210, 215 [, 234, ...]

Explanation: Taking the prime numbers to begin: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47. Then $A(n+1) = A(n) + p_m$, where m is the sum of digits of $A(n)$.

Method: Take differences between terms, notice they're all primes, then determine how the primes chosen are related to the terms.

(β) Answer: 2, 2, 1, 1 [, 2, 1, 1, 1, 2, 2, ...]

Explanation: The Linus Sequence (after a cartoon) (A006345). Choose the next digit from $\{1, 2\}$ so as to minimize the length of the longest repeated substring of the sequence ending with the new character. [Tricky to express unambiguously and concisely!] Method: To find the next term, try each of $\{1, 2\}$ in turn, then working from the end of the sequence find the length of the longest duplicated substring (e.g., in '1 2 1 1 2 2' it is '2', with length 1 using up the last 2 digits; in '1 2 1 1 2 1' it is '1 2 1' with length 3 using up all 6 digits) and choose the shorter of the two options.

(γ) Answer: 728, 5053, 40341, 362914, [3628855, ...]

Explanation: Sum of Fibonacci and Factorials

Method: Sequence of three cubes intended to mislead. Try subtracting cubes—can't explain the '2, 3' at start; try subtracting factorials—and then recognise the residue as Fibonacci numbers.

(δ) Answer: 246, 285, 320, 362

Explanation: Sum of the squares of the integers from 0 to n, expressed in base $n+1$.

Method: Notice the low digits used in the first terms, the sudden jump to three-digit numbers and the uneven distribution of the numbers and deduce that it's likely to be a base-n sequence. Convert the numbers back to decimal, take differences, and the sequence becomes obvious.

(ϵ) Answer: 4, 4, 4, 4

Explanation: The National Anthem (CCDBCDEEFEDCDCBCGGGG)

Method: The '-1' is the clue which rules out the 'properties of n^{th} integer' types of answers normal for sequences of small numbers. Also a clue in the title of the problem.

Answer to Problem 1

$7 \times 3942 = 73 \times 9 \times 42 = 27594$

Mark scheme: 1/6 for each symbol given the correct digit.

Answer to Problem 2

(a)	(xvii)	(k)	(xxvi)	(u)	(v)
(b)	(xxiv)	(l)	(xiv)	(v)	(viii)
(c)	(xix)	(m)	(xviii)	(w)	(xii)
(d)	(iii)	(n)	(iv)	(x)	(ix)
(e)	(xxi)	(o)	(xxv)	(y)	(vi)
(f)	(xxii)	(p)	(x)	(z)	(xi)
(g)	(xxvii)	(q)	(xx)	(aa)	(xvi)
(h)	(xxix)	(r)	(xxviii)	(bb)	(vii)
(i)	(i)	(s)	(xv)	(cc)	(xiii)
(j)	(ii)	(t)	(xxiii)		

Mark scheme: 1/58 for each correct answer, the total being doubled if self-consistent (i.e., no number given for more than one name).

Answer to Problem 3

1/60

Mark scheme: 1 for correct answer, 0 for 0 or below or 1/15 or above; otherwise linear in two pieces.

Answer to Problem 4

$\pi^2/6$ (in base 14; the second line is a table of squares in base 14).

Method: There are 14 distinct digits present (plus a 'decimal' point), and 14 numbers in the bottom row. Guess base 14 and that the bottom row is a table of squares; this gives enough of the digits for a good idea of the value of the first row, which is $\pi^2/6 = \zeta(2)$.

[The Z in the title is a **capital** Zeta; the stone is 14 pounds. There's also an allusion to the dimensions of the monolith in *2001* $(1:4:9:\cdots$, in the book).]

Mark scheme: 1 for correct answer, 1/2 for an approximately correct decimal not interpreted as $\pi^2/6$.

Answer to Problem 5

$(5\sqrt{3}/36)$ (the density and circumradius are red herrings; all that's relevant is the centre of mass, which is over the centre of the triangle).

Mark scheme: 1 for correct answer, 0 for $\sqrt{3}/8$ or below or $\sqrt{3}/4$ or above; otherwise linear in two pieces.

Answer to Problem 6

E60.

Method: Recognise that 75025 is a Fibonacci number and it's easy. Otherwise use trial and error for small numbers. Can be proved by induction for the general case.

Optimal Algorithm: At each stage, choose a subset of the remaining addresses of a size equal to the largest Fibonacci number that fits. Net cost as each smaller Fibonacci number gets used is 2.50.

75025 is the 24th Fibonacci number, so answer is $24 \times 2.50 = 60$. (e.g. $8 = 5 + 3$; If Eugene is in the 5, cost is 2.50; if in the 3, cost is 5 etc. with the smaller answer neatly costing twice the larger answer, but the larger answer being one 2.50 step further away from the smaller one.) Other algorithms, such as halving, risk costing E5 at every step; $2^{17} = 131072$; $17 \times 5 =$ E85.

Mark scheme: $2^{-|\text{difference}|}$.

Answer to Problem 7

$2000/2 \times 2000 \times (1000!)/((500!)^2) \times 1000/(2^{1000})$

Method: Calculate from one vertex then multiply up by $(2000/2)$. $\text{Diameter}^{1000} = 1000$; $\text{Radius}^{1000} = 1000/(2^{1000})$. Combinatorics and Sums of trigonometric functions eg using imaginary numbers/exponential function. $\binom{1000}{500} = (1000!)/((500!)^2)$.

Mark scheme: 1 for correct answer, 0 for wrong answer.

Answer to Problem 8

$(13 \times 9 + 4)/(9999999999999 \times 9) = 121/89999999999991$

$142857 \times 3 = 428571$; $285714 \times 3 = 857142$; $142857142857 \times 3 = 428571428571$; $285714285714 \times 3 = 857142857142$.

Mark scheme: 1/2 for correct fraction, 1/8 for each correct sum (but any such 1/8 cancelled by a wrong sum).

Method:

Multiplication by 1: $1 \times 1 = 1$; $11 \times 1 = 11$; $111 \times 1 = 111$; $\ldots$; $1111111111111 \times 1 = 1111111111111$; $2 \times 1 = 2$; $22 \times 1 = 22$;...$9 \times 1 = 9$;...; total of 13×9 correct calculations.

Multiplication by 2: No solutions. Answer must have same number of digits, so first digit can't exceed 4; first digit gets to end; answer must be even so first digit must be even; so first digit must be 2 or 4. Write the original number as $(2 \times 10^A + B)$ or $(4 \times 10^A + B)$ Then $(2 \times 10^A + B) \times 2 = 10 \times B + 2$ so $4 \times 10^A - 2 = 8 \times B$ so $B = (2 \times 10^A - 1)/4$ which is not an integer since numerator is odd. Similarly for $(4 \times 10^A + B)$.

Multiplication by 3: Similar approach gives $B = (3 \times 10^A - 1)/7$. $B = 42857$; so $142857 \times 3 = 428571$; also $285714 \times 3 = 857142$; and $142857142857 \times 3 = 428571428571$; $285714285714 \times 3 = 857142857142$. Total of 4 solutions.

Multiplication by 4: First digit must be 2; $B = (8 \times 10^A - 2)/6$ which exceeds 10^A; but B must be an A-digit number so no solutions.

Multiplication by 5: Must begin with 1 in order to fit; but then last digit becomes 1, not divisible by 5 so no solutions.

Multiplication by 6: Must begin with 1 in order to fit; but then last digit becomes 1, not divisible by 6 so no solutions.

Multiplication by 7: Must start with 1; $B = (7 \times 10^A - 1)/3$ which exceeds 10^A; but B must be an A-digit number so no solutions.

Multiplication by 8: Must begin with 1 in order to fit; but then last digit becomes 1, not divisible by 8 so no solutions.

Multiplication by 9: Must start with 1; $B = (9 \times 10^A - 1)/1 > 10^A$ so no solutions.

Answer to Problem 9

5, 7, 8, 9, 10, 11. The one that's not so obvious is 8: the presence of an octagon in a tiling (not necessarily Archimedean) by regular polygons forces the whole tiling to be 8.8.4.

Mark scheme: 1/6 for each correct value, cancelled by $-1/6$ for each incorrect value.

The Archimedeans' anthem (from *Eureka* **1**) is:

All praise to Archimedes,
  Who weighed the royal hat,
Displacing quarts of h. and c.,
  Upon the bathroom mat.
For that unending decimal,
  We mortals know as π,
He found three-and-one-seventh,
  Was just a bit too high.

All praise to Arthur Eddington,
  Who proved I don't know what,
Except that ev'rything you think's,
  Exactly what it's not.
He knows what Albert Einstein's,
  Equations are about;
And that's where he has you and me,
  And Einstein, up the spout.

to the tune of Hymn 341, A. and M. [132 in A. and M. Revised]. [This anthem was written by Mr. Hope-Jones of Eton College, in 1938, when he was unable to come to speak to the Archimedeans. Metrical analysis by Philip Belben also available.]

Answer to Problem X

$1-(\pi/4)$

Mark scheme: piecewise linear, 0 at 0 and 1, 1 at correct answer.

Answer to Problem E

Option (e); marks following the mark scheme. For (b) of the second part, the correct answer was 3.07471281226; for (c) of the second part it was 0.45229885054. The first part was found somewhat confusing; "(b) or (e)" was also accepted as an answer.

The trick is to realise that there are more than 4 statements—otherwise answer (b) could also be true!

Mark scheme: 1/2 for first part if correct; $(1/6)2^{-(\text{difference}^2)}$ for each guessed score.

3 Results

The winners scored approximately 3.224 (out of 12); one pair scored 3.384 but included a previous winner so was disqualified from winning this time. The lowest score was 0.826. The prize for silliest answer was awarded for the answer $1+\ln(9/2)$ to question 3 (where clearly any answer greater than 1/3 would be absurd). Certificates were presented to all these four pairs commemorating their achievements. The background designs of the certificates for the winners, wooden spoon and silliest answer are respectively Figures 4(n), 4(l) and 2(a) of Joseph Myers, Tiling with Regular Star Polygons, *Eureka* **56** (March 2004), 20–27; for the properties of these tilings, see that article.