An inequality arising in graph minors

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Lemma 1 *Let* $p_0 = \frac{1}{3}\left(4 + \sqrt[3]{3\sqrt{33}-17} - \sqrt[3]{3\sqrt{33}+17}\right) = 0.45631\ldots$ *be the real root of* $x^3 - 4x^2 + 6x - 2 = 0$. *Let* $f_q(\alpha) = 1 - \alpha^2 - (1-\alpha)^2 + \alpha^2 q^{1/\alpha^2} + (1-\alpha)^2 q^{1/(1-\alpha)^2} - q$. *If* $0 < \alpha < 1$ *and* $0 \leq q < q_0 = 1 - p_0$, *then* $f_q(\alpha) > 0$. *Further, for* $0 \leq q < q_0$, *we have* $f_q(\frac{1}{100}) > 10^{-3}$.

Proof Observe first that for $\alpha = \frac{1}{2}$, $f_q(\alpha) = \frac{1}{2} + \frac{1}{2}q^4 - q$. Thus $f_q(\frac{1}{2}) > 0$ if and only if $q^4 - 2q + 1 = (q-1)(q^3+q^2+q-1) > 0$, which on $0 < q < 1$ occurs just when $q > q_0$. Note also that $f_q(0) = f_q(1) = 0$. Clearly, f_q is symmetrical about $\alpha = \frac{1}{2}$.

We have

$$\frac{\partial f}{\partial \alpha} = -4\alpha + 2 + q^{1/\alpha^2}(2\alpha - 2\alpha^{-1}\log q) - q^{1/(1-\alpha)^2}(2(1-\alpha) - 2(1-\alpha)^{-1}\log q)$$

and so

$$\frac{\partial^2 f}{\partial \alpha^2} = -4 + q^{1/\alpha^2}(2 - 2\alpha^{-2}\log q + 4\alpha^{-4}(\log q)^2) + q^{1/(1-\alpha)^2}(2 - 2(1-\alpha)^{-2}\log q + 4(1-\alpha)^{-4}(\log q)^2)$$

and all these functions are well-defined and continuous in α for any $0 \leq q < 1$.

The lemma may now be proved by numeric computation. The function $-r\log r$ on $[0,1]$ is zero at 0 and 1, and has a unique maximum at $r = e^{-1}$. The function $r(\log r)^2$ on $[0,1]$ is zero at 0 and 1, and has a unique maximum at $r = e^{-2}$. The second derivative above is composed of constants, q^{1/α^2}, these functions for $r = q^{1/\alpha^2}$, and corresponding functions of $1-\alpha$. Thus, given bounds on q and α, we may deduce bounds on the second partial derivative. To be precise, given $r_{\min} \leq r \leq r_{\max}$, we have

$$\min\{-r_{\min}\log r_{\min}, -r_{\max}\log r_{\max}\} \leq -r\log r \leq e^{-1}$$

if $r_{\min} \leq e^{-1} \leq r_{\max}$, and

$$\min\{-r_{\min}\log r_{\min}, -r_{\max}\log r_{\max}\} \leq -r\log r \leq \max\{-r_{\min}\log r_{\min}, -r_{\max}\log r_{\max}\}$$

otherwise; similarly, we have

$$\min\{r_{\min}(\log r_{\min})^2, r_{\max}(\log r_{\max})^2\} \leq r(\log r)^2 \leq 4e^{-2}$$

if $r_{\min} \leq e^{-2} \leq r_{\max}$, and

$$\min\{r_{\min}(\log r_{\min})^2, r_{\max}(\log r_{\max})^2\} \leq r(\log r)^2 \leq \max\{r_{\min}(\log r_{\min})^2, r_{\max}(\log r_{\max})^2\}$$

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otherwise. If we now have that $q_{\min} \leq q \leq q_{\max}$ and $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$, we consider these bounds applied to q^{1/α^2} and $q^{1/(1-\alpha)^2}$, which satisfy

$$q_{\min}^{1/\alpha_{\min}^2} \leq q^{1/\alpha^2} \leq q_{\max}^{1/\alpha_{\max}^2}$$

and

$$q_{\min}^{1/(1-\alpha_{\max})^2} \leq q^{1/(1-\alpha)^2} \leq q_{\max}^{1/(1-\alpha_{\min})^2}.$$

At $\alpha = \frac{1}{2}$, the first partial derivative is zero. For $0 \leq q \leq 0.4$, we prove the lemma by showing that the second partial derivative is negative for $0 \leq \alpha \leq \frac{1}{2}$. This may be done by considering the numerical bounds for four regions of (q, α) space, being $0 \leq \alpha \leq \frac{1}{4}$; $\frac{1}{4} \leq \alpha \leq \frac{3}{8}$; $\frac{3}{8} \leq \alpha \leq \frac{7}{16}$; and $\frac{7}{16} \leq \alpha \leq \frac{1}{2}$, each being for the whole range $0 < q \leq 0.4$. The bounds are as follows:

$q_{\min}$	$q_{\max}$	$\alpha_{\min}$	$\alpha_{\max}$	$\partial^2_{\min}$	$\partial^2_{\max}$
0	0.4	0	$\frac{1}{4}$	−4.000000	−0.298494
0	0.4	$\frac{1}{4}$	$\frac{3}{8}$	−4.000000	−0.529831
0	0.4	$\frac{3}{8}$	$\frac{7}{16}$	−4.000000	−0.390413
0	0.4	$\frac{7}{16}$	$\frac{1}{2}$	−4.000000	−0.101764

For $0.4 \leq q \leq 0.48$, determine the bounds on the second partial derivative for regions of $\frac{1}{32}$ in α. These then give bounds on the first partial derivative (working from $\frac{1}{2}$ to 0), which then bound $f_q(\alpha)$ itself, showing it to be greater than 0 for $0 < \alpha < 1$. Given bounds $\partial^2_{\min}[(r-1)/32, r/32]$ and $\partial^2_{\max}[(r-1)/32, r/32]$ on the second partial derivative for $(r-1)/32 \leq \alpha \leq r/32$, and bounds $\partial_{\min}(r/32)$ and $\partial_{\max}(r/32)$ on the first partial derivative for $\alpha = r/32$, we compute $\partial_{\min}((r-1)/32) = \partial_{\min}(r/32) - \partial^2_{\max}[(r-1)/32, r/32]/32$ and $\partial_{\max}((r-1)/32) = \partial_{\max}(r/32) - \partial^2_{\min}[(r-1)/32, r/32]/32$; and given these formulas, bounds on the first partial derivative for $(r-1)/32 \leq \alpha \leq r/32$ may be found as $\partial_{\min}[(r-1)/32, r/32] = \min\{\partial_{\min}((r-1)/32), \partial_{\min}(r/32)\}$ and $\partial_{\max}[(r-1)/32, r/32] = \max\{\partial_{\max}((r-1)/32), \partial_{\max}(r/32)\}$. Similarly, we then compute bounds on $f_q(\alpha)$, starting from $f_q(0) = 0$: we find successively that $f_{\min}((r+1)/32) = f_{\min}(r/32) + \partial_{\min}[r/32, (r+1)/32]/32$, $f_{\max}((r+1)/32) = f_{\max}(r/32) + \partial_{\max}[r/32, (r+1)/32]/32$; $f_{\min}[r/32, (r+1)/32] = \min\{f_{\min}(r/32), f_{\min}((r+1)/32)\}$ and $f_{\max}[r/32, (r+1)/32] = \max\{f_{\max}(r/32), f_{\max}((r+1)/32)\}$. The main bounds (from which the others may trivially be derived) are as follows:

$q_{\min}$	$q_{\max}$	$\alpha_{\min}$	$\alpha_{\max}$	$\partial^2_{\min}$	$\partial^2_{\max}$	$\partial_{\min}(\alpha_{\min})$	$\partial_{\max}(\alpha_{\min})$	$f_{\min}(\alpha_{\max})$	$f_{\max}(\alpha_{\max})$
0.4	0.48	$\frac{0}{32}$	$\frac{1}{32}$	-1.507705	-0.868127	0.169366	0.633552	0.004445	0.019798
0.4	0.48	$\frac{1}{32}$	$\frac{2}{32}$	-1.460135	-0.816579	0.142237	0.586436	0.008092	0.038125
0.4	0.48	$\frac{2}{32}$	$\frac{3}{32}$	-1.409836	-0.765012	0.116719	0.540807	0.010993	0.055025
0.4	0.48	$\frac{3}{32}$	$\frac{4}{32}$	-1.365339	-0.714791	0.092813	0.496749	0.013195	0.070548
0.4	0.48	$\frac{4}{32}$	$\frac{5}{32}$	-1.327749	-0.668062	0.070475	0.454083	0.014745	0.084738
0.4	0.48	$\frac{5}{32}$	$\frac{6}{32}$	-1.291636	-0.628018	0.049599	0.412590	0.015682	0.097632
0.4	0.48	$\frac{6}{32}$	$\frac{7}{32}$	-1.258632	-0.601366	0.029973	0.372227	0.016031	0.109264
0.4	0.48	$\frac{7}{32}$	$\frac{8}{32}$	-1.230934	-0.590225	0.011180	0.332895	0.015804	0.119667
0.4	0.48	$\frac{8}{32}$	$\frac{9}{32}$	-1.211104	-0.580139	-0.007264	0.294428	0.015011	0.128868
0.4	0.48	$\frac{9}{32}$	$\frac{10}{32}$	-1.198827	-0.535069	-0.025394	0.256581	0.013695	0.136886
0.4	0.48	$\frac{10}{32}$	$\frac{11}{32}$	-1.182100	-0.415480	-0.042115	0.219118	0.011973	0.143733
0.4	0.48	$\frac{11}{32}$	$\frac{12}{32}$	-1.140395	-0.193886	-0.055098	0.182177	0.010062	0.149426
0.4	0.48	$\frac{12}{32}$	$\frac{13}{32}$	-1.182572	0.098926	-0.061157	0.146540	0.008150	0.154006
0.4	0.48	$\frac{13}{32}$	$\frac{14}{32}$	-1.186989	0.410351	-0.058066	0.109584	0.006336	0.157430
0.4	0.48	$\frac{14}{32}$	$\frac{15}{32}$	-1.168614	0.657096	-0.045242	0.072491	0.004922	0.159696
0.4	0.48	$\frac{15}{32}$	$\frac{16}{32}$	-1.151091	0.790659	-0.024708	0.035972	0.004150	0.160820

Finally, on the regions $0.48 \le q \le 0.5$ and $0.5 \le q \le 0.55$ separately, we show that f has the expected shape so that it cannot be zero anywhere if $f_q(\frac{1}{2}) > 0$. That is, we show (using steps of $\frac{1}{64}$ in α) that, for some $0 < \alpha_0 < \frac{1}{2}$, we have the second partial derivative negative for $\alpha \le \alpha_0$, and that the first partial derivative is negative for $\alpha_0 \le \alpha < \frac{1}{2}$. The first partial derivative is positive at $\alpha = 0$, and so we have a single minimum at $\alpha = \frac{1}{2}$. The main bounds are as follows:

$q_{\min}$	$q_{\max}$	$\alpha_{\min}$	$\alpha_{\max}$	$\partial^2_{\min}$	$\partial^2_{\max}$	$\partial_{\min}(\alpha_{\min})$	$\partial_{\max}(\alpha_{\min})$
0.48	0.5	$\frac{0}{64}$	$\frac{1}{64}$	−1.408230	−1.213716	0.239120	0.403462
0.48	0.5	$\frac{1}{64}$	$\frac{2}{64}$	−1.384527	−1.187189	0.220156	0.381458
0.48	0.5	$\frac{2}{64}$	$\frac{3}{64}$	−1.360114	−1.160157	0.201606	0.359825
0.48	0.5	$\frac{3}{64}$	$\frac{4}{64}$	−1.335026	−1.132696	0.183479	0.338573
0.48	0.5	$\frac{4}{64}$	$\frac{5}{64}$	−1.309306	−1.104899	0.165780	0.317714
0.48	0.5	$\frac{5}{64}$	$\frac{6}{64}$	−1.283016	−1.076882	0.148516	0.297256
0.48	0.5	$\frac{6}{64}$	$\frac{7}{64}$	−1.256231	−1.048782	0.131690	0.277209
0.48	0.5	$\frac{7}{64}$	$\frac{8}{64}$	−1.229047	−1.020766	0.115303	0.257580
0.48	0.5	$\frac{8}{64}$	$\frac{9}{64}$	−1.201584	−0.993031	0.099353	0.238376
0.48	0.5	$\frac{9}{64}$	$\frac{10}{64}$	−1.173989	−0.965464	0.083837	0.219601
0.48	0.5	$\frac{10}{64}$	$\frac{11}{64}$	−1.147909	−0.937643	0.068752	0.201258
0.48	0.5	$\frac{11}{64}$	$\frac{12}{64}$	−1.123369	−0.909815	0.054101	0.183322
0.48	0.5	$\frac{12}{64}$	$\frac{13}{64}$	−1.099521	−0.882967	0.039885	0.165769
0.48	0.5	$\frac{13}{64}$	$\frac{14}{64}$	−1.076647	−0.857503	0.026089	0.148589
0.48	0.5	$\frac{14}{64}$	$\frac{15}{64}$	−1.054954	−0.832877	0.012690	0.131766
0.48	0.5	$\frac{15}{64}$	$\frac{16}{64}$	−1.034245	−0.807094	−0.000323	0.115283
0.48	0.5	$\frac{16}{64}$	$\frac{17}{64}$	−1.013303	−0.776203	−0.012934	0.099123
0.48	0.5	$\frac{17}{64}$	$\frac{18}{64}$	−0.989288	−0.734603	−0.025062	0.083290
0.48	0.5	$\frac{18}{64}$	$\frac{19}{64}$	−0.957649	−0.676263	−0.036540	0.067832
0.48	0.5	$\frac{19}{64}$	$\frac{20}{64}$	−0.912850	−0.596326	−0.047107	0.052869
0.48	0.5	$\frac{20}{64}$	$\frac{21}{64}$	−0.849667	−0.492464	−0.056425	0.038606
0.48	0.5	$\frac{21}{64}$	$\frac{22}{64}$	−0.764558	−0.365542	−0.064119	0.025330
0.48	0.5	$\frac{22}{64}$	$\frac{23}{64}$	−0.656591	−0.219517	−0.069831	0.013383
0.48	0.5	$\frac{23}{64}$	$\frac{24}{64}$	−0.527732	−0.060788	−0.073261	0.003124
0.48	0.5	$\frac{24}{64}$	$\frac{25}{64}$	−0.382543	0.102733	−0.074211	−0.005122
0.48	0.5	$\frac{25}{64}$	$\frac{26}{64}$	−0.227493	0.266095	−0.072606	−0.011099
0.48	0.5	$\frac{26}{64}$	$\frac{27}{64}$	−0.088946	0.429870	−0.068448	−0.014653
0.48	0.5	$\frac{27}{64}$	$\frac{28}{64}$	0.035108	0.585497	−0.061731	−0.016043
0.48	0.5	$\frac{28}{64}$	$\frac{29}{64}$	0.143961	0.718313	−0.052583	−0.015495
0.48	0.5	$\frac{29}{64}$	$\frac{30}{64}$	0.231379	0.822463	−0.041359	−0.013245
0.48	0.5	$\frac{30}{64}$	$\frac{31}{64}$	0.292463	0.894044	−0.028508	−0.009630
0.48	0.5	$\frac{31}{64}$	$\frac{32}{64}$	0.323857	0.930475	−0.014539	−0.005060

$q_{\min}$	$q_{\max}$	$\alpha_{\min}$	$\alpha_{\max}$	$\partial^2_{\min}$	$\partial^2_{\max}$	$\partial_{\min}(\alpha_{\min})$	$\partial_{\max}(\alpha_{\min})$
0.5	0.55	$\frac{0}{64}$	$\frac{1}{64}$	−1.578014	−1.199428	0.121452	0.428178
0.5	0.55	$\frac{1}{64}$	$\frac{2}{64}$	−1.557062	−1.172509	0.102711	0.403521
0.5	0.55	$\frac{2}{64}$	$\frac{3}{64}$	−1.535210	−1.144971	0.084390	0.379192
0.5	0.55	$\frac{3}{64}$	$\frac{4}{64}$	−1.512451	−1.116871	0.066500	0.355205
0.5	0.55	$\frac{4}{64}$	$\frac{5}{64}$	−1.488781	−1.088285	0.049049	0.331573
0.5	0.55	$\frac{5}{64}$	$\frac{6}{64}$	−1.464206	−1.059305	0.032045	0.308311
0.5	0.55	$\frac{6}{64}$	$\frac{7}{64}$	−1.438742	−1.030045	0.015493	0.285432
0.5	0.55	$\frac{7}{64}$	$\frac{8}{64}$	−1.412418	−1.000644	−0.000601	0.262952
0.5	0.55	$\frac{8}{64}$	$\frac{9}{64}$	−1.385274	−0.971269	−0.016237	0.240883
0.5	0.55	$\frac{9}{64}$	$\frac{10}{64}$	−1.357373	−0.942121	−0.031413	0.219238
0.5	0.55	$\frac{10}{64}$	$\frac{11}{64}$	−1.328795	−0.913393	−0.046133	0.198029
0.5	0.55	$\frac{11}{64}$	$\frac{12}{64}$	−1.299646	−0.884563	−0.060405	0.177267
0.5	0.55	$\frac{12}{64}$	$\frac{13}{64}$	−1.270059	−0.855292	−0.074226	0.156960
0.5	0.55	$\frac{13}{64}$	$\frac{14}{64}$	−1.245036	−0.824604	−0.087590	0.137115
0.5	0.55	$\frac{14}{64}$	$\frac{15}{64}$	−1.220577	−0.789540	−0.100475	0.117661
0.5	0.55	$\frac{15}{64}$	$\frac{16}{64}$	−1.195318	−0.744631	−0.112811	0.098590
0.5	0.55	$\frac{16}{64}$	$\frac{17}{64}$	−1.167012	−0.683183	−0.124446	0.079913
0.5	0.55	$\frac{17}{64}$	$\frac{18}{64}$	−1.131434	−0.597688	−0.135121	0.061678
0.5	0.55	$\frac{18}{64}$	$\frac{19}{64}$	−1.082693	−0.482643	−0.144460	0.044000
0.5	0.55	$\frac{19}{64}$	$\frac{20}{64}$	−1.014446	−0.336355	−0.152001	0.027083
0.5	0.55	$\frac{20}{64}$	$\frac{21}{64}$	−0.921518	−0.161436	−0.157257	0.011232
0.5	0.55	$\frac{21}{64}$	$\frac{22}{64}$	−0.801255	0.035759	−0.159779	−0.003167
0.5	0.55	$\frac{22}{64}$	$\frac{23}{64}$	−0.654199	0.246316	−0.159220	−0.015686
0.5	0.55	$\frac{23}{64}$	$\frac{24}{64}$	−0.484007	0.460107	−0.155372	−0.025908
0.5	0.55	$\frac{24}{64}$	$\frac{25}{64}$	−0.296809	0.666970	−0.148182	−0.033471
0.5	0.55	$\frac{25}{64}$	$\frac{26}{64}$	−0.100312	0.857566	−0.137761	−0.038109
0.5	0.55	$\frac{26}{64}$	$\frac{27}{64}$	0.097117	1.026778	−0.124362	−0.039676
0.5	0.55	$\frac{27}{64}$	$\frac{28}{64}$	0.287213	1.177975	−0.108318	−0.038158
0.5	0.55	$\frac{28}{64}$	$\frac{29}{64}$	0.420332	1.308571	−0.089912	−0.033671
0.5	0.55	$\frac{29}{64}$	$\frac{30}{64}$	0.520205	1.416793	−0.069466	−0.027103
0.5	0.55	$\frac{30}{64}$	$\frac{31}{64}$	0.589474	1.494816	−0.047328	−0.018975
0.5	0.55	$\frac{31}{64}$	$\frac{32}{64}$	0.624918	1.534205	−0.023972	−0.009764

It remains to show the lower bound on $f_q(\frac{1}{100})$. For this, observe

$$\begin{aligned}
\frac{\partial f}{\partial \alpha} &= -4\alpha + 2 + q^{1/\alpha^2}(2\alpha - 2\alpha^{-1}\log q) - q^{1/(1-\alpha)^2}(2(1-\alpha) - 2(1-\alpha)^{-1}\log q) \\
&\geq -4\alpha + 2 - q^{1/(1-\alpha)^2}(2(1-\alpha) - 2(1-\alpha)^{-1}\log q) \\
&\geq -4\alpha + 2 - q(2(1-\alpha) - 2(1-\alpha)^{-1}\log q) \\
&\geq -4\alpha + 2 - 2q + 2(1-\alpha)^{-1} q \log q \\
&\geq -4\alpha + 2 - 2q_0 - 2(1-\alpha)^{-1} e^{-1}.
\end{aligned}$$

This last expression is decreasing in α, and for $\alpha = \frac{1}{100}$ this value is positive, so $f_q(\frac{1}{100}) \geq \frac{1}{100}(-4 \times \frac{1}{100} + 2 - 2q_0 - 2(1 - \frac{1}{100})^{-1}e^{-1}) > 10^{-3}$. $\square$