Tiling with Regular Star Polygons

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The Archimedean tilings (Figure 1) and polyhedra will be familiar to many readers. They have the property that the tiles of the tiling, or the faces of the polyhedron, are regular polygons, and that the vertices form a single orbit under the symmetries of the tiling or polyhedron. (Grünbaum and Shephard [1] use *Archimedean*, in relation to tilings, to refer to the sequence of polygons at each vertex being the same, and *uniform* to refer to the vertices forming a single orbit. These describe the same set of tilings, but the sets of k-uniform tilings (those with k orbits of vertices) and k-Archimedean tilings (those with k different types of vertices) differ for $k > 1$. I do not make this distinction in this article, but use the term *uniform* to avoid ambiguity. In relation to polyhedra, the distinction made between these terms is different.)

The Archimedean polyhedra were attributed to Archimedes by Pappus [3, 4], although the work of Archimedes on them has not survived; the tilings may have been named by analogy. The first surviving systematic account of either the tilings or the polyhedra seems to be that of Kepler [5, 6]. The literature on the 2-uniform and 3-uniform tilings is discussed by Grünbaum and Shephard [1]; the k-uniform tilings for $k \leq 6$ are presented by Galebach [15].

Given these tilings and polyhedra, for centuries people have generalised in different ways (for example, through changing the definitions of tilings and polyhedra, through changing the permissible tiles and faces, or through considering analogous concepts in higher dimensions). Some of these generalisations have yielded more aesthetically pleasing results than others. One form of generalisation, considered by Kepler, is allowing star polygons. Two different types of regular star polygons may be considered. One, the modern version, treats a star n-gon as a polygon with n edges, which intersect each other; only the n endpoints of those edges are considered as corners of the polygon, and not the points of intersection of the sides.[1] The notion of 'tilings' with such polygons is not very clear, but it has been considered thoroughly [7]; many of these tilings are not especially aesthetically pleasing because of the density of the crossing lines that make up the edges of the polygons, and a single drawing can represent multiple distinct tilings. When polyhedra with such polygons as faces are considered, the set of uniform polyhedra [8, 9, 10, 11, 12] appears; these are rather more attractive; some readers may have seen the author's models of some of these polyhedra on the Archimedeans' Societies Fair stand in 2002. Kepler considered regular polyhedra with this notion, finding the small and great stellated dodecahedra but not the great dodecahedron or great icosahedron which were later found by Poinsot [13].

The other type of regular star n-gon (guided more by aesthetics than by mathematical generalisation) is a nonconvex $2n$-gon with equal sides and alternating angles; *n points* of angle α (with $0 < \alpha < (n-2)\pi/n$) and *n dents* of angle $2(n-1)\pi/n - \alpha$; we denote this polygon n_α. When considering tilings, Kepler used this notion; he drew various patches of tilings using such polygons, and mentioned various such tilings found in the course of enumerating the uniform tilings with regular convex polygons. However, he never made it clear exactly which polygons and tilings were allowed. This type of regular star polygon yields more attractive tilings than the modern more mathematical type of star polygon (which yields tilings rather too densely cluttered with lines). It would be natural mathematically to consider

[1] With this version, a polygon is considered regular if its symmetries act transitively on the pairs (vertex, edge incident with that vertex). Infinite polygons, aperiogons and zigzags, may be allowed; when they are, [7] notes that their enumeration of tilings is only conjectural.

polygons with equal sides and alternating angles, convex or nonconvex, but given the aesthetic and historical motivation we do not do so here.

Grünbaum and Shephard [2] made the first attempt at enumeration of such tilings (and so some sort of completion of Kepler's enumeration) under definite rules, attempting to find uniform or k-uniform tilings with regular polygons and any n_α star polygons. In [1] they adjusted the definitions used, so that the *vertices* of the tiling are only those points where three or more tiles meet; if a dent of a star is filled entirely by the corner of one other polygon, that is a corner of the polygons but not a vertex of the tiling. They also consider tilings that are not edge-to-edge: where the polygons involved may have different edge lengths, and some vertices are in the middle of edges. They presented drawings of uniform tilings involving star polygons, which they conjectured show all such tilings, giving as an exercise proving that their list of uniform tilings involving star polygons which are not edge-to-edge is complete, and another exercise asking whether there were any other (edge-to-edge) uniform tilings involving star polygons. Apart from this work and that of Kepler, such tilings do not seem to have been considered in the mathematical literature, although some are shown in [14].

When I attempted those exercises in 1993, it turned out, however, that those lists were not complete; there are three uniform tilings, one of them not edge-to-edge, which are missing from their lists, shown in Figures 2(a), 4(l) and 4(n). These additional tilings were used as designs for certificates presented to those receiving awards in the 2001 Problems Drive [16]. This article presents the full enumeration, with an outline of how it may be verified.

First we consider how the uniform tilings without star polygons may be enumerated. If k regular polygons with $n_1, \ldots, n_k$ edges respectively meet at a vertex, we must have

$$\sum_{i=1}^{k} \frac{n_i - 2}{n_i} = 2.$$

Clearly $3 \leq k \leq 6$ and for each k it is easy to determine the finitely many solutions. There are 17 possible choices of the n_i, where different orders are not counted as distinct; where different cyclic orders are counted as distinct (but the reversal of an order is counted as the same as that order), this yields 21 possible species of vertices. Some of these cannot occur in any tiling by regular polygons at all; for example, $3 \,.\, 7 \,.\, 42$ is the only possibility involving a heptagon, and this would mean that triangles and 42-gons must alternate around the heptagon, which is impossible since 7 is odd. This leaves 15 species that can occur in tilings by regular polygons. Of these, 11 yield the uniform tilings shown in Figure 1; it turns out that each yields a unique uniform tiling.[2] The vertex $4 \,.\, 8^2$ can only appear in the uniform tiling it generates, and not in any other tiling by regular polygons, since it is the only one of the 15 species containing an octagon; the others can appear in k-uniform tilings for suitable k (and, for sufficiently large k, all species can appear together in one tiling).

Suppose now we consider tilings involving regular star polygons. In addition to ordinary regular polygons, a vertex of such a tiling may have points and dents of star polygons. We only consider uniform tilings, so all vertices are alike. Observe that no vertex can have two dents present, and two star points cannot be adjacent at a vertex. Also, since the tiling is supposed to contain some star polygon, it is easy to see that some vertex, and so all vertices, must have a star point. (For, if any star point is not a vertex, it fills a dent of a second star;

[2]Properly, it is necessary to show that each of the tilings in Figure 1 actually exists, since it is easy to draw what look like tilings by regular polygons but are actually fakes with polygons that are not exactly regular; examples of such drawings may be found in children's colouring books. It is not quite trivial that 'local' existence of the tilings implies global existence, but we do not discuss existence proofs further here.

but then the points of that star on either side of the filled dent must lie at vertices of the tiling.)

First we dispose of the tilings that are not edge-to-edge. This means that a vertex lies part way along the edge of some polygon. The vertex cannot have a dent, or two adjacent star points, but it must have a star point. Thus it has present either two ordinary regular polygons (one a triangle, the other a triangle, square or pentagon), with at least one star point (on one side, or between the polygons), or one regular polygon with star points on either or both sides. Bearing in mind that there cannot be a vertex at a dent, a careful analysis of cases (which the reader is encouraged to verify; it is convenient to start by showing that the vertex must lie on the edge of an ordinary regular polygon, not a star) shows that the possible tilings are those of Figure 2. The new one (used in the Problems Drive certificate for silliest answer) is Figure 2(a).

Having found those tilings, we need now only consider edge-to-edge tilings, in which all polygons will have the same edge length. In general the analysis of these is more systematic than that of the tilings that are not edge-to-edge. Because it essentially consists of analysis of many cases, most of the details are not presented here but are left to the reader, who will need to draw a large number of little diagrams for the various cases.

It is convenient to separate the cases where some dent lies at a vertex of the tiling (so, since the tiling is uniform, all vertices have a dent) from those where no dent is a vertex. If some dent is a vertex, it cannot be filled entirely by points of stars, so the polygons in the dent are k (for some integer k) or $3 \,.\, 3$, $3 \,.\, 4$ or $3 \,.\, 5$, and each case is considered in turn, yielding the tilings in Figure 3 (four of which actually show an example of an infinite family of tilings).

Now suppose that no dent is a vertex. Considering the possible vertex figures, clearly no two star points can be adjacent; since no dent is a vertex, no point can lie between two regular polygons with different numbers of edges; and if a point of a star lies between two regular polygons with the same number of edges, their next vertices must fill its dents exactly. This means that there must be two adjacent regular polygons with the same number of vertices, separated by the point of a star, since we are only looking for tilings which do involve star polygons. This leads to the table (Table 1) of cases for the sequence of regular polygons (ignoring the star polygons). The new tilings are Figure 4(n) (used for the certificate for the winners) and Figure 4(l) (used for the certificate for the wooden spoon).

By way of example, consider the case k^2 ($k \geq 3$). If there is a single star point at the vertex, say s_α, we have $\alpha = 4\pi/k$ and $2(s-1)\pi/s - \alpha = \pi + 2\pi/k$, so $2/s + 6/k = 1$. There is a combinatorial constraint that 3 divides k, and the integer solutions yield Figures 4(e), (f) and (g). If there are two star points at the vertex, say s_α and t_β, both dents are filled by the k-gon vertex, so $2(s-1)\pi/s - \alpha = \pi + 2\pi/k = 2(t-1)\pi/t - \beta$; thus $\alpha = (1 - 2/s - 2/k)\pi$ and $\beta = (1 - 2/t - 2/k)\pi$, yielding $1 = 4/k + 1/s + 1/t$. Combinatorially, k is even, and if $s \neq t$ then 4 divides k; the solutions subject to these constraints yield Figures 4(h) to (k).

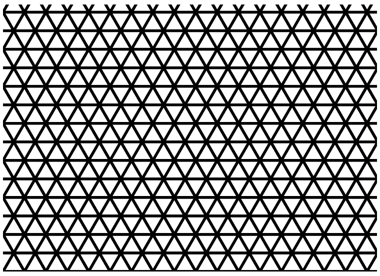
A natural extension of this work would be to enumerate 2-uniform edge-to-edge tilings by regular polygons and regular star polygons. I have done some work towards this, but completing such an enumeration by hand would be substantially time-consuming and error-prone. As Kepler's diagrams include various examples of k-uniform tilings (and of small patches that can plausibly be extended to such tilings) such enumeration could be seen as continuing the systematic completion of Kepler's work. It would be interesting to develop a sufficiently systematic method of finding k-uniform tilings involving star polygons that such tilings could be searched for by computer. Even for tilings not involving star polygons, more efficient enumeration algorithms might be able to extend the enumeration far beyond that of [15]. A deeper problem would be to attempt to gain some understanding of the asymptotic behaviour of the number of k-uniform tilings of any variety.

Another direction would be to attempt to determine which polygons can occur at all in edge-to-edge tilings by regular polygons and regular star polygons, with no uniformity conditions. For example, can regular n-gons or n-stars with $n > 18$ occur? Can regular heptagons occur?

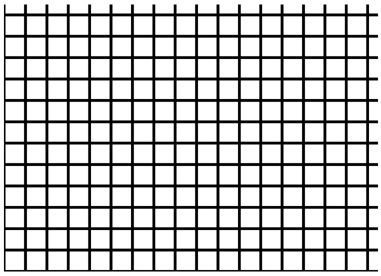
When considering tilings that are not edge-to-edge, Grünbaum and Shephard present as a research exercise determining the 2-uniform tilings, not involving star polygons, with the additional constraint that the tilings be equitransitive (i.e., that the symmetries act transitively on each congruence class of tiles). This problem could be attempted with or without that constraint, or with star polygons allowed. An attempt could also be made at a search algorithm for such tilings that could be implemented on computer.

Vertex sequence	Tilings (Figure 4)
3^5	(a)
$3^4 . 4$	(b)
$3^4 . 5$	None
3^4	None
$3^3 . k \quad (k \geq 4)$	None
3^3	None
$3^2 . k \quad (k \geq 4)$	None
$3^2 . 5^2$	None
$3^2 . a . b \quad (4 \leq a \leq 5 \leq b, a < b)$	(c), (d)
$3^2 . 4^2$	None
$k^2 \quad (k \geq 3)$	(e) to (k)
$3 . k^2 \quad (k \geq 4)$	(l), (m)
$3 . 4^3$	(n)
$3 . 4^2 . 5$	None
4^3	(o)
$4^2 . k \quad (k \geq 5)$	None
$4 . k^2 \quad (5 \leq k \leq 7)$	(p)
5^3	(q)
$5^2 . k \quad (6 \leq k \leq 9)$	None
$5 . 6^2$	None

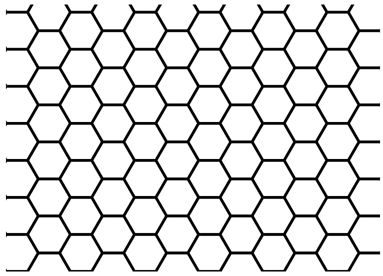
Table 1: Cases where no dent is a vertex



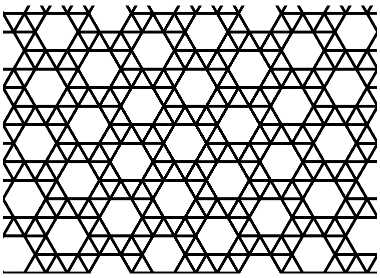
(a) (3^6)



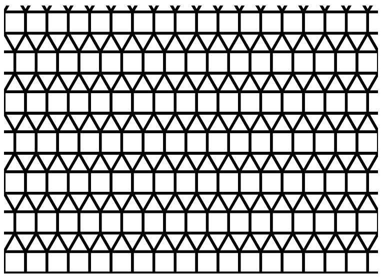
(b) (4^4)



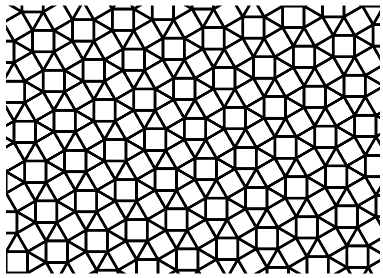
(c) (6^3)



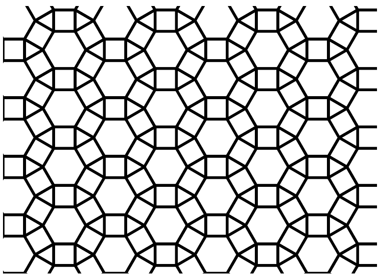
(d) $(3^4 . 6)$



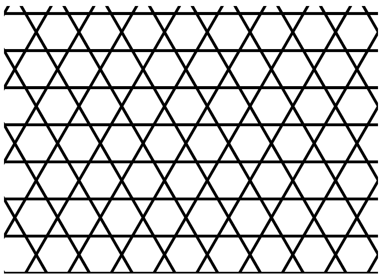
(e) $(3^3 . 4^2)$



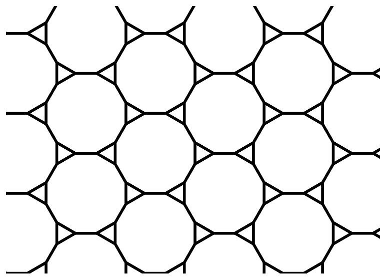
(f) $(3^2 . 4 . 3 . 4)$



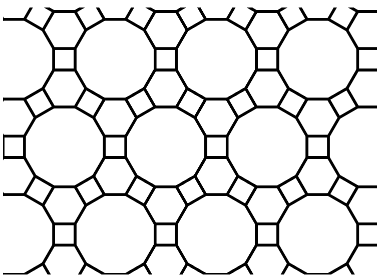
(g) $(3 . 4 . 6 . 4)$



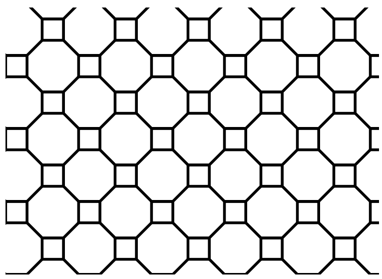
(h) $(3 . 6 . 3 . 6)$



(i) $(3 . 12^2)$

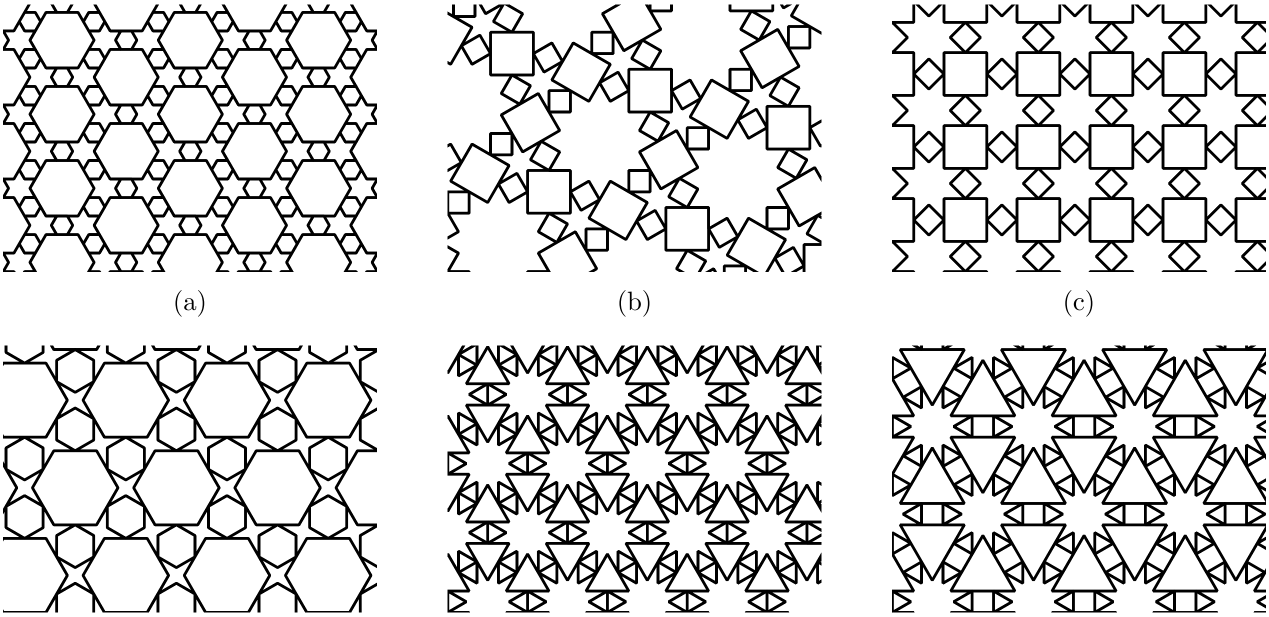


(j) $(4 . 6 . 12)$



(k) $(4 . 8^2)$

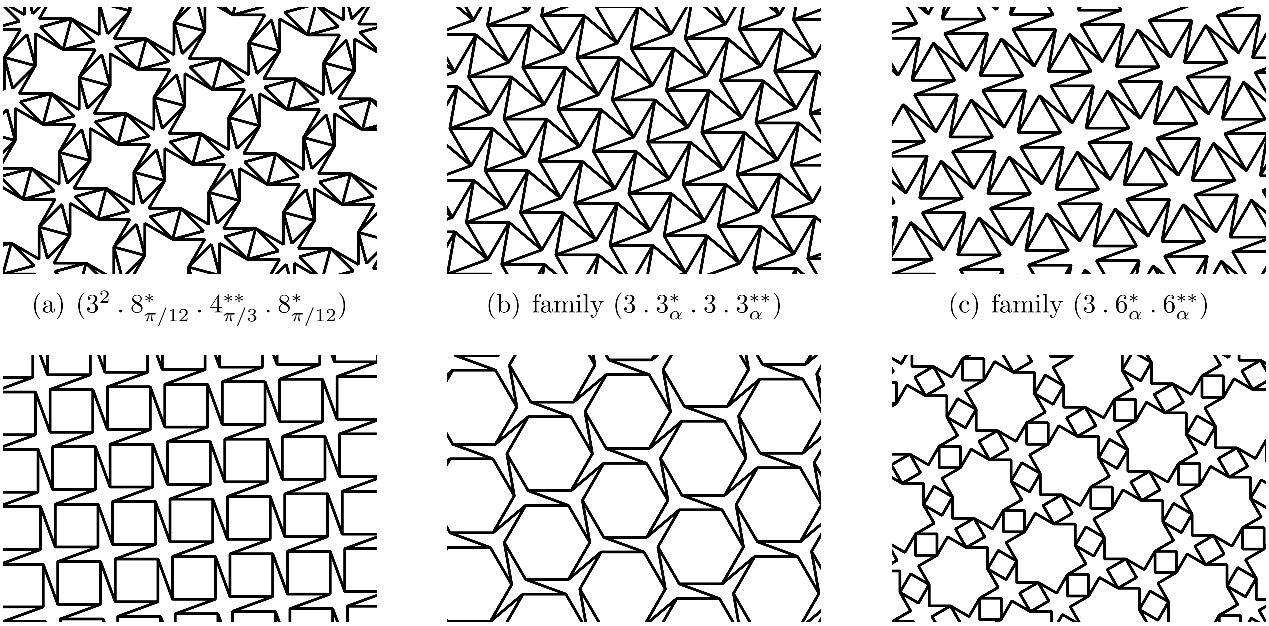
Figure 1: Archimedean tilings



(a) (b) (c)

(d) (e) (f)

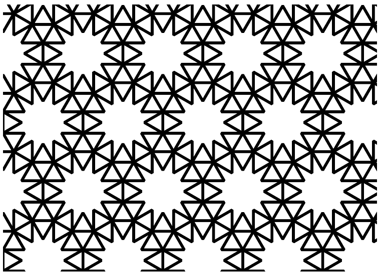
Figure 2: Uniform tilings involving star polygons which are not edge-to-edge



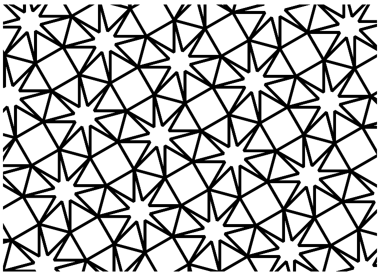
(a) $(3^2 . 8^*_{\pi/12} . 4^{**}_{\pi/3} . 8^*_{\pi/12})$ (b) family $(3 . 3^*_\alpha . 3 . 3^{**}_\alpha)$ (c) family $(3 . 6^*_\alpha . 6^{**}_\alpha)$

(d) family $(4 . 4^*_\alpha . 4^{**}_\alpha)$ (e) family $(6 . 3^*_\alpha . 3^{**}_\alpha)$ (f) $(4 . 6^*_{\pi/6} . 6^{**}_{\pi/2} . 6^*_{\pi/6})$

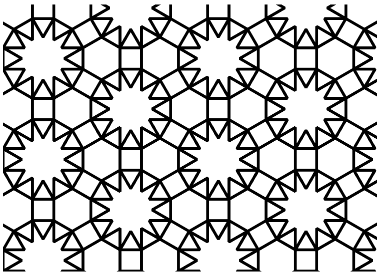
Figure 3: Uniform tilings in which some dent is a vertex



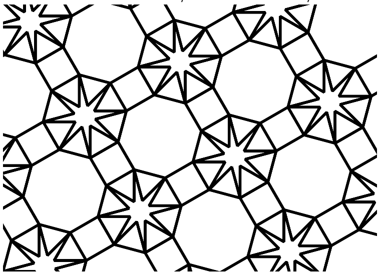
(a) $(3^3 . 12^*_{\pi/6} . 3^2 . 12^*_{\pi/6})$



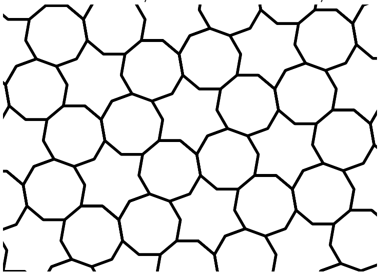
(b) $(3^2 . 8^*_{\pi/12} . 3 . 4 . 3 . 8^*_{\pi/12})$



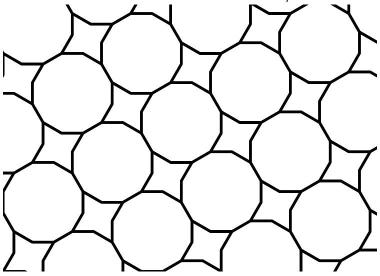
(c) $(3 . 4 . 6 . 3 . 12^*_{\pi/6})$



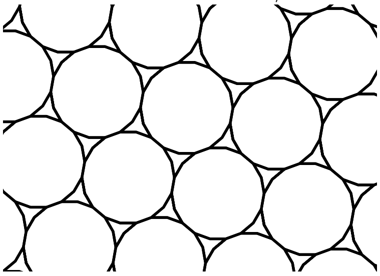
(d) $(3 . 4 . 8 . 3 . 8^*_{\pi/12})$



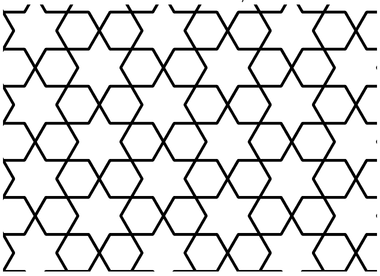
(e) $(9^2 . 6^*_{4\pi/9})$



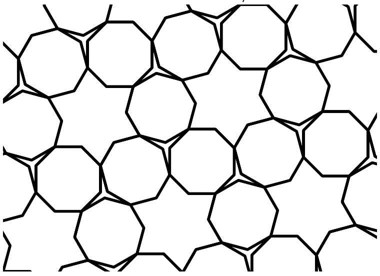
(f) $(12^2 . 4^*_{\pi/3})$



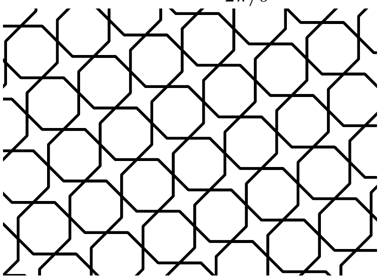
(g) $(18^2 . 3^*_{2\pi/9})$



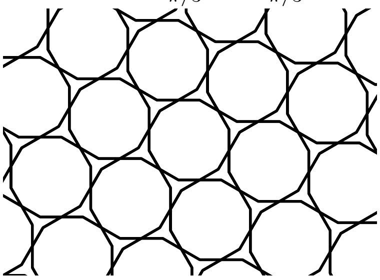
(h) $(6 . 6^*_{\pi/3} . 6 . 6^*_{\pi/3})$



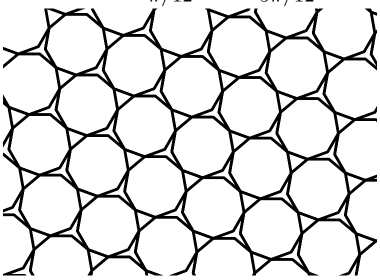
(i) $(8 . 3^*_{\pi/12} . 8 . 6^*_{5\pi/12})$



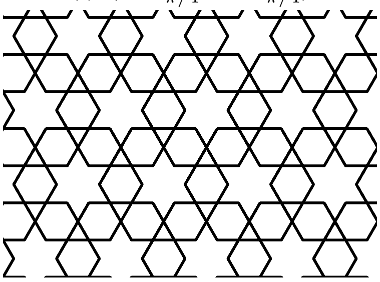
(j) $(8 . 4^*_{\pi/4} . 8 . 4^*_{\pi/4})$



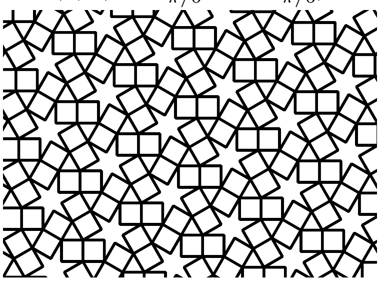
(k) $(12 . 3^*_{\pi/6} . 12 . 3^*_{\pi/6})$



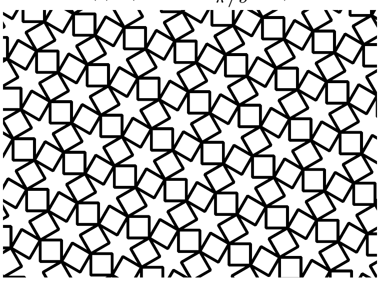
(l) $(3 . 9 . 3^*_{\pi/9} . 9)$



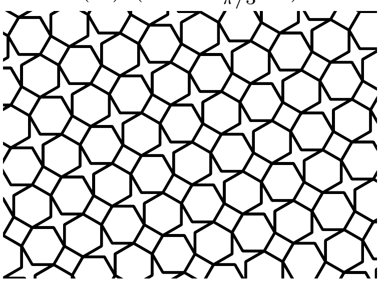
(m) $(3 . 6 . 6^*_{\pi/3} . 6)$



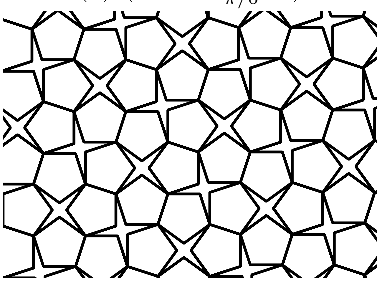
(n) $(3 . 4^2 . 6^*_{\pi/6} . 4)$



(o) $(4 . 6^*_{\pi/6} . 4 . 6^*_{\pi/6} . 4 . 6^*_{\pi/6})$



(p) $(4 . 6 . 4^*_{\pi/6} . 6)$



(q) $(5^2 . 4^*_{\pi/10} . 5 . 4^*_{\pi/10})$

Figure 4: Uniform tilings in which no dent is a vertex

References

[1] Branko Grünbaum and G. C. Shephard, *Tilings and Patterns*, W. H. Freeman and Company, New York, 1987

[2] Branko Grünbaum and Geoffrey C. Shephard, *Tilings by Regular Polygons*, Mathematics Magazine **50** (1977), 227–247 and **51** (1978), 205–206

[3] Pappus Alexandrinus (edited by Friedrich Hultsch), *Pappi Alexandrini collectionis quae supersunt e libris manuscriptis edidit, latina interpretatione et commentariis instruxit F. Hultsch*, 1876

[4] Ivor Thomas, *Selections illustrating the history of Greek Mathematics II*, Heinemann, London, 1941 (Relevant part of Pappus translated under Archimedes)

[5] Johannes Kepler, *Harmonices mundi libri v*, Lincii Austriae, 1619

[6] E. J. Field, A. M. Duncan and J. V. Field, eds., *The Harmony of the World by Johannes Kepler; translated into English with an introduction and notes*, American Philosophical Society, 1997

[7] Branko Grünbaum, J. C. P. Miller and G. C. Shephard, *Tilings with Hollow Tiles*, in *The Geometric Vein—The Coxeter Festschrift*, Chandler Davis, Branko Grünbaum and F. A Sherk, eds., Springer-Verlag, New York, 1981, 17–64

[8] H. S. M. Coxeter, M. S. Longuet-Higgens and J. C. P. Miller, *Uniform polyhedra*, Philosophical Transactions of the Royal Society of London, Series A **246** (1954), 401–450

[9] J. Skilling, *The complete set of uniform polyhedra*, Philosophical Transactions of the Royal Society of London, Series A **278** (1975), 111–135

[10] M. J. Wenninger, *Polyhedron Models*, Cambridge University Press, Cambridge, 1971

[11] Zvi Har'El, *Uniform Solution for Uniform Polyhedra*, Geometriae Dedicata **47** (1993), 57–110

[12] Peter W. Messer, *Closed-Form Expressions for Uniform Polyhedra and Their Duals*, Discrete and Computational Geometry **27** (2002), 353–375

[13] L. Poinsot, *Mémoire Sur les Polygones et les Polyèdres*, Journal de l'École Polytechnique **10** (1810), 16–48

[14] Lyndon Baker, NRICH Logoland, October 2000 and November 2000

`http://nrich.maths.org/mathsf/journalf/oct00/logoland.html`
`http://nrich.maths.org/mathsf/journalf/nov00/logoland.html`

[15] Brian L. Galebach, n-Uniform Tilings. 2002–3

`http://ProbabilitySports.com/tilings.html`

[16] Alasdair Kergon and Joseph Myers, Problems Drive 2001

`http://www.srcf.ucam.org/~jsm28/publications/2001/problems-drive.ps`