The extremal function for unbalanced bipartite minors

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Abstract

We consider the question of what average degree forces a graph to have a $K_{s,t}$ minor, for s fixed and t sufficiently large. In the case of $s = 2$, we show that if t is sufficiently large and G is a graph with more than $\frac{t+1}{2}(|G| - 1)$ edges then G has a $K_{2,t}$ minor. This result is best possible for $|G| \equiv 1 \pmod{t}$.

Key words: Graph minors, Extremal problems

1 Introduction

As usual, define a graph H to be a *minor* of a graph G (writing $H \prec G$) if H can be obtained from G by a series of vertex and edge deletions and edge contractions; or, equivalently, if there are disjoint subsets $W_u \subseteq V(G)$, for $u \in V(H)$, such that all $G[W_u]$ are connected and, for all $uv \in E(H)$, there is an edge in G between W_u and W_v. Write $\mathcal{G}(n, p)$ for a random graph on n vertices where each edge is independent and has probability p of being present.

Mader [1] showed that, for any t, any graph G with a sufficiently large (in terms of t) average degree must have a K_t minor. Bollobás, Catlin and Erdős [2] determined what order of complete minor occurs in a random graph, $n/\sqrt{\log_{1/q} n}$ for a $\mathcal{G}(n, 1 - q)$ random graph. Fernandez de la Vega [3] observed that this showed that random graphs are good examples of graphs with high average degree but no large complete minor, and that it implied that the necessary average degree to force a K_t minor was not just a linear function of t.

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Kostochka [4,5] and Thomason [6] showed that random graphs are within a constant factor of being optimal. The exact extremal function was then determined by Thomason [7]: the average degree that forces a K_t minor is $(1 + o(1))\alpha t\sqrt{\log t}$ for an explicitly determined constant α. The random graphs achieving this extremum are graphs of a certain order and a fixed density $\rho = 0.71533\ldots$; in [8] it is shown that all extrema have quasi-random properties.

In [9] we study the extremal problem for general H; that is, the problem of determining what average degree forces an H minor. It turns out that, for almost all H, random graphs with the same density $\rho = 0.71533\ldots$ provide extremal examples and determine the extremal function, even for quite sparse H. We now wish to find examples of H for which the extremal graphs are not random with density ρ (or essentially disjoint unions or random graphs with that density). In this paper, we consider the case of $K_{s,t}$ where s is fixed and t is large. In this case, the extremal graphs are no longer random with density ρ, and are only random at all if we admit complete graphs as random graphs of density 1. For example, in the trivial case where the graph we wish to avoid as a minor is the star $K_{1,t}$, the extremal graphs avoiding this minor are the union of disjoint K_t graphs; in general, graphs with no $K_{s,t}$ minor may be found as the union of many K_{t+s-1} graphs with $s-1$ common vertices (see Theorem 4), which are better than random graphs. (The general result for when $K_{s,t}$ minors occur in $\mathcal{G}(n, 1-q)$ random graphs, found in [10], is as follows. For positive integers n, t and real $0 < q < 1$, put

$$\ell_n(t) = \left\lfloor \frac{n/t + 1}{2} \right\rfloor,$$

and put

$$s_{n,q}(t) = \frac{\ell_n(t)(n - \ell_n(t)t)}{\log_{1/q} n}.$$

It will turn out that $\ell_n(t)$ is the optimal order of the parts of the minor on the t-side, with those on the s-side being of order $(\log_{1/q} n)/\ell_n(t)$. The value of $\ell_n(t)$ arises from maximising $(n - \ell t)\ell$ for integer ℓ. The largest s for which we have a $K_{s,t}$ minor, or a $K_s + \overline{K_t}$ minor, if $t \geq n/2\sqrt{\log_{1/q} n}$, is essentially $s_{n,q}(t)$.)

In many cases, it seems that $K_s + \overline{K_t}$ minors occur just when $K_{s,t}$ minors do; in [10] we see this to be the case for minors in random graphs. For this reason, we also discuss $K_s + \overline{K_t}$ minors in this paper, although without proving results for them.

Of course, an average degree $O\left(t\sqrt{\log t}\right)$ forces a K_{s+t} minor, and so a $K_{s,t}$ minor. However, a better bound on the average degree that forces such a minor would be desirable. I conjecture the following:

Conjecture 1 *Let s be a positive integer. Then there exists a constant C such that, for all positive integers t, if G has average degree at least Ct, then $K_{s,t} \prec G$.*

In this paper we determine the exact average degree that forces a $K_{2,t}$ minor, for t sufficiently large. The main result of the paper, which is best possible for $|G| \equiv 1 \pmod{t}$, is Theorem 2.

Theorem 2 *Let $t > 10^{29}$ be a positive integer. Let G be a graph with more than $\frac{t+1}{2}(|G|-1)$ edges. Then G has a $K_{2,t}$ minor.*

This is the simplest nontrivial case of the extremal problem for $K_{s,t}$ minors with s fixed. The results are only stated and proved for $K_{2,t}$ minors, but many of the arguments are more general and indications are given where appropriate of how they may be adapted to $K_s+\overline{K_t}$ minors. These indications only describe how the arguments given could be generalised; in some places, additional arguments not given here would also be needed. The arguments for $K_{2,t}$ minors might appear more complicated than one would expect; this complexity seems necessary, although much of it is only needed to achieve a best possible average degree of $t+1$; an average degree of $t+2$ can be achieved without many of the special cases; in particular, none of the special cases in Lemma 9 are needed for such a weaker result. We return to $K_{s,t}$ and $K_s + \overline{K_t}$ minors at the end of the paper.

2 Simple bounds

We observed that the star is a trivial case of a complete bipartite minor. We state the obvious bounds for star minors formally here.

Theorem 3 *Let $t \geq 1$ be some integer. If a graph has average degree greater than $t-1$ then it has a $K_{1,t}$ minor, but there exist arbitrarily large graphs with average degree $t-1$ and no $K_{1,t}$ minor.*

PROOF. For the first part, if a graph has average degree greater than $t-1$ then it has a vertex v of degree at least t, and v together with t of its neighbours provides a $K_{1,t}$ subgraph, which is a minor. For the second part, consider graphs that are the union of arbitrarily many disjoint K_t subgraphs.

The following construction provides a general lower bound, which turns out to be the correct bound for $s = 2$.

Theorem 4 *Let $2 \le s \le t$ be integers and let $k \ge 1$ be an integer. Let G be the graph on $kt + s - 1$ vertices that is the union of k graphs K_{t+s-1}, there being $s - 1$ vertices shared among all those K_{t+s-1} and all the other vertices of G being in exactly one K_{t+s-1}. Then G does not contain a $K_{s,t}$ minor.*

PROOF. Suppose that G has a $K_{s,t}$ minor, so that there are disjoint subsets $V_1, V_2, \ldots, V_s, W_1, W_2, \ldots, W_t$ of $V(G)$ such that all $G[V_i]$ and $G[W_j]$ are connected and there is an edge from V_i to W_j for all i and j.

Because there are only $s - 1$ vertices of G shared among all the K_{t+s-1}, at least one of the V_i does not contain any of those vertices; likewise, since $s \le t$, at least one of the W_j does not contain any of those vertices. There must be an edge between any such V_i and W_j, so all such V_i and W_j lie entirely within the same t vertices that are in just one of the K_{t+s-1} making up G. All other V_i and W_j must have a vertex in the $s-1$ shared vertices; but this implies that all V_i and W_j have at least one vertex within the same K_{t+s-1}, a contradiction since the V_i and W_j are disjoint and there are $t + s$ of them.

Corollary 5 *Let $2 \le s \le t$ be integers. For any $\epsilon > 0$, there exist arbitrarily large graphs G with average degree at least $t + 2s - 3 - \epsilon$ and no $K_{s,t}$ minor.*

PROOF. The graph G of Theorem 4 has

$$\begin{aligned} &k\left(\tfrac{1}{2}t(t-1) + t(s-1)\right) + \tfrac{1}{2}(s-1)(s-2) \\ &\qquad\qquad = k\left(\tfrac{1}{2}t(t+2s-3)\right) + \tfrac{1}{2}(s-1)(s-2) \end{aligned}$$

edges. This gives an average degree of

$$\frac{kt(t+2s-3) + (s-1)(s-2)}{kt+s-1} = t + 2s - 3 - \frac{(s-1)(t+s-1)}{kt+s-1},$$

which tends to $t + 2s - 3$ from below as $k \to \infty$.

3 Small graphs

In this section we prove Theorem 2 in the case where $|G|$ is not much bigger than t.

In general we consider a graph G, of order $t+d$, where $d < \frac{1}{11}t^{1/4}$, and suppose that this graph has more than $\frac{t+1}{2}(|G|-1)$ edges. Clearly we need only consider $d \geq 2$. We then show that there are disjoint subsets A and B of $V(G)$, such that $G[A]$ and $G[B]$ are connected, A has at least $t+\frac{d}{2}$ neighbours outside A, and B has at least $t+\frac{d}{2}$ neighbours outside B. Then A and B provide one half of the minor, and the intersection of the sets of neighbours provides the other half. Each of A and B will in fact consist of a single vertex, or a pair of neighbouring vertices.

The case of $d \leq 3$ turns out to be a special case, which we readily dispose of:

Lemma 6 *Let t be a positive integer. Let G be a graph of order $t+2$ or $t+3$ with more than $\frac{t+1}{2}(|G|-1)$ edges. Then G has a $K_{2,t}$ minor.*

PROOF. If G is of order $t+2$, then it has two vertices of degree $t+1$; for otherwise, $e(G) \leq \frac{1}{2}(t|G|+1) = \frac{1}{2}(t^2+2t+1) = \frac{t+1}{2}(|G|-1)$, a contradiction. Those two vertices have t common neighbours, yielding our minor.

Now suppose G is of order $t+3$, so it has at least $\frac{1}{2}(t+1)(t+2)+1 = \frac{1}{2}(t^2+3t+4)$ edges. If G has a vertex x of degree $t+2$, then it has some other vertex y of degree at least $t+1$; for otherwise, $e(G) \leq \frac{1}{2}(t|G|+2) = \frac{1}{2}(t^2+3t+2)$, a contradiction. Then x and y have t common neighbours. Otherwise, we see in the same way that G must have at least four vertices of degree $t+1$. If any two of these are nonneighbours, then they have $t+1$ common neighbours. Suppose then that there are exactly k vertices of degree $t+1$ and none of greater degree; let those vertices be $x_1, x_2, \ldots, x_k$. Each of them has $t+2-k$ neighbours in the rest of the graph, so exactly one nonneighbour in the rest of the graph; let the nonneighbour of x_i be y_i. No two y_i are the same, since if $y_i = y_j$ then x_i and x_j would have t common neighbours. If there were an edge between y_i and y_j then contracting that edge would yield our minor, one half having the vertices x_i and x_j and the other half having all the other vertices of the new graph. Thus there are no edges among the y_i. The degrees of all vertices of G add up to at least $t|G|+4$; all vertices other than the x_i have degree at most t; we have $d(y_i) \leq t+2-k$ for $1 \leq i \leq k$, and so $d(x_i)+d(y_i) \leq 2t+3-k \leq 2t$ for $1 \leq i \leq k$, yielding a contradiction.

For larger d, we find A and B separately, finding both of them by the same method; our results will show that, given a set $X \subset V(G)$ with $|X| \leq 2$, there is a subset $Y \subset V(G) \setminus X$ with $G[Y]$ connected, $|Y| \leq 2$ and Y having at least $t+\frac{d}{2}$ neighbours outside Y. This can then be applied with X empty, to find A, then with $X = A$, to find B.

Lemma 7 *Let t and d be positive integers. Let G be a graph of order $t+d$. Let $X \subset V(G)$ with $|X| \leq 2$. Let the maximum degree (in G) of any vertex in*

$V(G) \setminus X$ be $t+s$, where $0 \le s < \frac{d}{2}$. Let j be an integer with $0 \le j \le s$, and write $d_h = \lceil \frac{d}{2} - j \rceil$. Suppose that there does not exist a subset $Y \subset V(G) \setminus X$ with $G[Y]$ connected, $|Y| \le 2$ and Y having at least $t+\frac{d}{2}$ neighbours outside Y. Then G has at most $\frac{1}{2}(t+d-1)(t+1) + \frac{1}{2}[t(d_h+s+j-d+1)+d^2+j^2-2dj-1+jd_h+js-2d_h]$ edges.

PROOF. Let v be some vertex not in X with degree at least $t+j$. Let A be a set of $t+j$ neighbours of v, and let B be the set of the remaining $d-j-1$ vertices of G.

Within B, there are at most $\binom{d-j-1}{2} = \frac{1}{2}(d^2+j^2-2dj-3d+3j+2)$ edges. From v to the rest of the graph there are at most $t+s$ edges. It remains to maximise the number of edges within A, plus the number from A to B. If we write $d_A(x)$ for the number of edges within A from a vertex $x \in A$, and $d_B(x)$ for the number of edges to B, then we need to maximise $\sum_{x\in A} d_B(x) + \frac{1}{2}d_A(x)$. Every vertex x in $A \setminus X$ has $d_B(x) \le d_h$ (since if $x \in A \setminus X$ had $\frac{d}{2} - j + 1$ neighbours in B, we could take $Y = \{v, x\}$). Every vertex x in $A \setminus X$ has $d_A(x) + d_B(x) \le t+s-1$; adding, we have $d_A(x) + 2d_B(x) \le t+s-1+d_h$, so $d_B(x) + \frac{1}{2}d_A(x) \le d_h + \frac{1}{2}(t+s-1-d_h)$, for vertices $x \in A \setminus X$. For any vertices $x \in A \cap X$, we know only that $d_B(x) + \frac{1}{2}d_A(x) \le |B| + \frac{1}{2}(|A|-1) = d-j-1+\frac{1}{2}(t+j-1)$. Note that $\sum_{x\in A} d_B(x) + \frac{1}{2}d_A(x)$ will be maximised if $|A \cap X| = 2$.

Thus, we have

$$
\begin{aligned}
e(G) &\le \tfrac{1}{2}(d^2+j^2-2dj-3d+3j+2) + (t+s) \\
&\quad + (t+j-2)\left(d_h + \tfrac{1}{2}(t+s-1-d_h)\right) \\
&\quad + 2\left(d-j-1+\tfrac{1}{2}(t+j-1)\right) \\
&= \tfrac{1}{2}[d^2+j^2-2dj-3d+3j+2+(2t+2s) \\
&\qquad + (t+j-2)(d_h+t+s-1) + (4d-2j-6+2t)] \\
&= \tfrac{1}{2}[d^2+j^2-2dj+d+j-4+4t+2s \\
&\qquad + (t+j-2)(d_h+t+s-1)] \\
&= \tfrac{1}{2}[d^2+j^2-2dj+d+j-4+4t+2s \\
&\qquad + \left(t^2+t(d_h+s+j-3)+(j-2)(d_h+s-1)\right)] \\
&= \tfrac{1}{2}[t^2+t(d_h+s+j+1) \\
&\qquad + d^2+j^2-2dj+d-2+jd_h+js-2d_h] \\
&= \tfrac{1}{2}(t+d-1)(t+1) \\
&\quad + \tfrac{1}{2}[t(d_h+s+j-d+1)+d^2+j^2-2dj-1+jd_h+js-2d_h].
\end{aligned}
$$

Corollary 8 *Let t and d be positive integers, with $4 \le d < \sqrt{t}$. Let G be a graph of order $t+d$ with more than $\frac{t+1}{2}(t+d-1)$ edges. Let $X \subset V(G)$*

with $|X| \leq 2$. Let the maximum degree (in G) of any vertex in $V(G) \setminus X$ be $t+s$. Then, if $s > \frac{d-1}{2}$ or $s < \frac{d-3}{2}$, $V(G)$ has a subset $Y \subset V(G) \setminus X$ with $G[Y]$ connected, $|Y| \leq 2$ and Y having at least $t + \frac{d}{2}$ neighbours outside Y.

PROOF. If $s > \frac{d-1}{2}$, then $s \geq \frac{d}{2}$, and Y can be a single vertex with degree $t+s$. If $s < 0$, we would have $e(G) \leq \frac{1}{2}(t(|G|-1) + 2(d-1)) = \frac{1}{2}(t(t+d-1) + 2d-2) = \frac{1}{2}((t+1)(t+d-1) + d - 1 - t)$, a contradiction. Thus we have $0 \leq s < \frac{d-3}{2}$.

Suppose for a contradiction that there is no such Y. If $s \geq 1$, put $j = 1$ in Lemma 7. If d is even, we have $s \leq \frac{d}{2} - 2$ and $d_h = \frac{d}{2} - 1$; if d is odd, we have $s \leq \frac{d}{2} - \frac{5}{2}$ and $d_h = \frac{d}{2} - \frac{1}{2}$. In either case, $d_h + s \leq d - 3$. If $s = 0$, put $j = 0$. We then have $\frac{1}{2}(t+d-1)(t+1) + 1 \leq e(G) \leq \frac{1}{2}(t+d-1)(t+1) + \frac{1}{2}[t(d_h + s + 2 - d) + d^2 - 2d + s - d_h]$ if $s \geq 1$, and $\frac{1}{2}(t+d-1)(t+1) + 1 \leq e(G) \leq \frac{1}{2}(t+d-1)(t+1) + \frac{1}{2}[t(d_h - d + 1) + d^2 - 1 - 2d_h]$ if $s = 0$. If $s \geq 1$, we deduce that

$$
\begin{aligned}
0 &\leq t(d_h + s + 2 - d) + d^2 - 2d - 2 + s - d_h \\
&\leq -t + d^2.
\end{aligned}
$$

If $s = 0$, we likewise deduce that

$$
\begin{aligned}
0 &\leq t(d_h - d + 1) + d^2 - 3 - 2d_h \\
&\leq -t + d^2.
\end{aligned}
$$

This is a contradiction by the constraint on the value of d.

Lemma 9 *Let t and d be positive integers, with $4 \leq d < \frac{1}{11}t^{1/4}$. Let G be a graph of order $t+d$ with more than $\frac{t+1}{2}(|G|-1)$ edges. Let $X \subset V(G)$ with $|X| \leq 2$. Let the maximum degree (in G) of any vertex in $V(G) \setminus X$ be $t+s$, where $\frac{d-3}{2} \leq s \leq \frac{d-1}{2}$. Then either G has a $K_{2,t}$ minor or $V(G)$ has a subset $Y \subset V(G) \setminus X$ with $G[Y]$ connected, $|Y| \leq 2$ and Y having at least $t + \frac{d}{2}$ neighbours outside Y.*

PROOF. We work as in the proof of Lemma 7, taking $j = s$. Let v, A and B be as in that proof. If $s = \frac{d-3}{2}$, we have $d_h = 2$; otherwise we have $d_h = 1$. Note that, while $d_B(x) + \frac{1}{2}d_A(x)$ is maximised if $d_B(x) = d_h$ and $d_A(x) = t+s-1-d_h$, if s is too large then $d_A(x) \geq t$ and v and x have t common neighbours, giving a $K_{2,t}$ minor. Write $c = 2s - (d-3)$, so $d = 2s+3-c$. Say a vertex $x \in A \setminus X$ is *good* if $d_B(x) + \frac{1}{2}d_A(x) \leq \frac{1}{2}(t+s-c)$, *poor* if $d_B(x) + \frac{1}{2}d_A(x) = \frac{1}{2}(t+s-c+1)$ and *bad* if $d_B(x) + \frac{1}{2}d_A(x) > \frac{1}{2}(t+s-c+1)$.

If $s = \frac{d-1}{2}$, we have $c = 2$. Considering the two possible values for $d_B(x)$, we see that if $s \geq 3$ there can be no bad or poor vertices, but if $s = 2$ (so $d = 5$) there can be no bad vertices but there can be poor vertices with $d_B(x) = 1$ and $d_A(x) = t + s - 3 = t - 1$. If $s = \frac{d-2}{2}$, we have $c = 1$; if $s \geq 2$ there can be no bad or poor vertices, but if $s = 1$ (so $d = 4$) there can be no bad vertices but there can be poor vertices with $d_B(x) = 1$ and $d_A(x) = t + s - 2 = t - 1$. Finally, if $s = \frac{d-3}{2}$, and so $c = 0$, again there can be no bad vertices, and if $s \geq 3$ there can be no poor vertices, but if $s = 2$ (so $d = 7$) there can be poor vertices with $d_B(x) = 2$ and $d_A(x) = t + s - 3 = t - 1$ and if $s = 1$ (so $d = 5$) there can be poor vertices with $d_B(x) = 2$ and $d_A(x) = t + s - 3 = t - 2$.

First suppose that there are at least $\frac{1}{60}\sqrt{t}$ good vertices; this will hold in particular when all vertices are good, which always occurs except in the four cases given above when there may be poor vertices. Supposing there is no $K_{2,t}$ minor, and no Y with the property of the lemma, we maximise the number of edges in the graph. The number of edges within B is at most $\binom{d-s-1}{2} = \binom{s+2-c}{2}$; the number from v to the rest of the graph is $t+s$; and $|B| = d-s-1 = s+2-c$; so we have

$$
\begin{aligned}
e(G) \leq & \binom{s+2-c}{2} + (t+s) + \tfrac{1}{2}(t+s-2)(t+s-c+1) - \tfrac{1}{2}\left(\tfrac{1}{60}\sqrt{t}\right) \\
& \quad + 2\left(s+2-c+\tfrac{1}{2}(t+s-1)\right) \\
&= \tfrac{1}{2}[(s+2-c)(s+1-c) + (2t+2s) + (t+s-2)(t+s-c+1) \\
& \quad - \tfrac{1}{60}\sqrt{t} + (2t+6s+6-4c)] \\
&= \tfrac{1}{2}[t^2 + dt - \tfrac{1}{60}\sqrt{t} + (s+2-c)(s+1-c) \\
& \quad + 8s + (s-2)(s-c+1) + 6 - 4c] \\
&= \tfrac{1}{2}(t+1)(t+d-1) \\
& \quad + \tfrac{1}{2}[(c-2-2s) - \tfrac{1}{60}\sqrt{t} + (s^2+3s-2cs+2+c^2-3c) \\
& \qquad + 8s + (s^2-s-cs-2+2c) + 6 - 4c] \\
&= \tfrac{1}{2}(t+1)(t+d-1) + \tfrac{1}{2}[-\tfrac{1}{60}\sqrt{t} + 2s^2 + 8s - 3cs + c^2 - 4c + 4].
\end{aligned}
$$

Since $e(G) > \frac{1}{2}(t+1)(t+d-1)$, we have

$$
\begin{aligned}
\tfrac{1}{60}\sqrt{t} &< 2s^2 + 8s - 3cs + c^2 - 4c + 4 \\
&< d^2/2 + d/4 + 8 \\
&< 2d^2,
\end{aligned}
$$

so $\sqrt{t} < 120d^2$. But since $d < \frac{1}{11}t^{1/4}$ we have $\sqrt{t} > 121d^2$, a contradiction.

It remains to consider the case where there are fewer than $\frac{1}{60}\sqrt{t}$ good vertices. Note that we have $t > 44^4 = 3748096$. Let A_G be the subset of A consisting of all vertices that are either good or in X; we then have $|A_G| \leq \frac{1}{60}\sqrt{t}+2 < \frac{1}{56}\sqrt{t}$. All other vertices of A are poor.

In each of the four cases enumerated above where there can be poor vertices, the poor vertices all have the same d_A and d_B values, and B is of a small constant size (2, 3 or 4 depending on the case). The cases are as follows:

- $d = 4$, $|A| = t + 1$, $d_A = t - 1$, $|B| = 2$, $d_B = 1$.
- $d = 5$, $|A| = t + 2$, $d_A = t - 1$, $|B| = 2$, $d_B = 1$.
- $d = 5$, $|A| = t + 1$, $d_A = t - 2$, $|B| = 3$, $d_B = 2$.
- $d = 7$, $|A| = t + 2$, $d_A = t - 1$, $|B| = 4$, $d_B = 2$.

In all these cases, $d_A + d_B \geq t$. We find a minor in one of two ways. First, if poor vertices x, $y \in A$ have $\Gamma_B(x) = \Gamma_B(y)$, then let one part of the minor be y and another be $\{v, x\}$. If x and y are not neighbours then those parts of the minor have at least t common neighbours; in the case where $d = 7$ and $d_A + d_B = t + 1$, those parts have t common neighbours even if x and y are neighbours. Second, we try to find poor vertices x_1, y_1, x_2, $y_2 \in A$ such that $\Gamma_B(x_1) = \Gamma_B(y_1)$ and $\Gamma_B(x_2) = \Gamma_B(y_2)$, such that x_1 and x_2 are neighbours, and y_1 and y_2 are neighbours, and $\{x_1, x_2\}$ and $\{y_1, y_2\}$ have t common neighbours so may be taken as the parts of our minor.

The simplest case to consider is that of $d = 7$. Here we only need two poor vertices with the same neighbours in B. We will have these as long as we have at least $7 = \binom{4}{2} + 1$ poor vertices, which we do by the bounds on t and $|A_G|$.

The next simplest case to consider is that of $d = 4$. Here each poor vertex has exactly one nonneighbour (which may or may not be poor) in A. By the above arguments, we may suppose that any two poor vertices that are not neighbours have different neighbours in B. If two poor vertices x and y are neighbours but share the same neighbour in B and the same nonneighbour in A, then they have $t - 2$ common neighbours in A, one common neighbour in B, and share the neighbour v, so we have our minor. Thus we may suppose that any element of A_G is a nonneighbour of at most two poor vertices. Thus there are at least 4 poor vertices which have poor nonneighbours, and so the poor vertices include at least 2 pairs of nonneighbours. Say that x_1 and y_2 are nonneighbours, and x_2 and y_1 are nonneighbours, where $\Gamma_B(x_1) = \Gamma_B(y_1) = \{b_1\}$, say, and $\Gamma_B(x_2) = \Gamma_B(y_2) = \{b_2\}$. Then $\{x_1, x_2\}$ and $\{y_1, y_2\}$ each have as neighbours the t other vertices of the graph, and we have our minor.

Now consider the case where $d = 5$ and $|B| = 2$. Let $B = \{b_1, b_2\}$ and write A_1 for the set of those poor vertices whose neighbour in B is b_1, and A_2 for the set of those poor vertices whose neighbour in B is b_2. Each poor vertex has exactly 2 nonneighbours in A; by the above arguments, all edges within A_1 are present, as are all edges within A_2. We will find x_1, $y_1 \in A_1$ and x_2, $y_2 \in A_2$

such that x_1 and x_2 are neighbours; y_1 and y_2 are neighbours; $\{x_1, x_2\}$ has as neighbours all but at most one vertex; $\{y_1, y_2\}$ has as neighbours all but at most one vertex; and, if both those sets do not have as neighbours all of G, their nonneighbours (which can only be in A_G) are the same. This will yield our minor.

To find those vertices, first observe that there can be no more than 2 poor vertices with any given pair of nonneighbours in A_G (since two such with the same neighbour in B would have t common neighbours). Thus there are fewer than $|A_G|^2$ poor vertices with both nonneighbours in A_G. Let A_1' be the result of removing all such vertices from A_1, and let A_2' be the result of removing all such vertices from A_2. At least one of these sets has order at least $5|A_G|$; without loss of generality suppose that is A_1'. Then there are at least 5 vertices in A_1' that, if they have any nonneighbour in A_G, have the same nonneighbour in A_G. Let those be A_1''.

Now take any vertex $x_1 \in A_1''$, and let y_2 be a nonneighbour of x_1 in A_2. If x_1 has any other nonneighbour z in A_2, remove from A_1'' all nonneighbours (at most 2) of z. Also remove from A_1'' any nonneighbour (other than x_1) of y_2. There is at least one vertex other than x_1 left in A_1''; let y_1 be such a vertex. Let x_2 be a nonneighbour in A_2 of y_1. Then x_1 and x_2 are neighbours, as are y_1 and y_2, and each pair has as neighbours v, all of B, all of A_1 and A_2, and all of A_G except possibly the single vertex allowed to be a nonneighbour of vertices in A_1''. Thus we have our minor.

Finally, consider the case where $d = 5$ and $|B| = 3$. Let $B = \{b_1, b_2, b_3\}$ and write A_{12} for the set of those poor vertices whose neighbours in B are $\{b_1, b_2\}$, and define A_{23} and A_{31} likewise. Each poor vertex has exactly 2 nonneighbours in A; by the above arguments, all edges within A_{12} are present, as are all edges within A_{23} and all edges within A_{31}. For some pair of those sets—say A_{12} and A_{23}—we will find x_{12}, $y_{12} \in A_{12}$ and x_{23}, $y_{23} \in A_{23}$ such that x_{12} and x_{23} are neighbours; y_{12} and y_{23} are neighbours; $\{x_{12}, x_{23}\}$ has as neighbours all but at most one vertex; $\{y_{12}, y_{23}\}$ has as neighbours all but at most one vertex; and, if both those sets do not have as neighbours all of G, their nonneighbours (which can only be in $A_G \cup A_{31}$) are the same. This will yield our minor.

To find those vertices, first observe that there can be no more than 3 poor vertices with any given pair of nonneighbours in A_G. Thus there are fewer than $\frac{3}{2}|A_G|^2$ poor vertices with both nonneighbours in A_G. Let A_{12}', A_{23}' and A_{31}' be the result of removing all such vertices from A_{12}, A_{23} and A_{31} respectively. There are at least $48|A_G|$ vertices left after this removal, so some one of those sets, without loss of generality A_{12}', has at least $16|A_G|$ vertices. Each vertex of A_{12}' has a nonneighbour in A_{23} or A_{31}, so without loss of generality suppose that at least $8|A_G|$ vertices have a nonneighbour in A_{23}, letting the set of such vertices be A_{12}''. Dividing up those vertices according to what nonneighbour, if any, they have in A_G, we arrive at a subset A'''_{12} with at least 8 vertices all of which have the same nonneighbour, if any, in A_G.

Now let x_{12} be any vertex of A'''_{12}, and let y_{23} be a nonneighbour of x_{12} in A_{23}. Remove from A'''_{12} the following vertices: any nonneighbour (other than x_{12}) of y_{23} (at most 1 vertex); any nonneighbours (other than x_{12}) of any nonneighbour (other than y_{23}) of x_{12} in A_{23} (at most 2 vertices); any vertex in A_{12} that shares a nonneighbour in A_{31} with y_{23} (at most 1 vertex); any nonneighbours in A_{12} of any vertex in A_{23} that shares a nonneighbour in A_{31} with x_{12} (at most 2 vertices). At least one vertex other than x_{12} remains in A'''_{12}. Let y_{12} be such a vertex, and let x_{23} be a nonneighbour of y_{12} in A_{23}. Then x_{12} and x_{23} are neighbours; y_{12} and y_{23} are neighbours; and each pair has as neighbours v, all of B, all of A_{12}, all of A_{23}, all of A_{31} (since we arranged that neither pair could share a nonneighbour in A_{31}), and all of A_G except possibly the one vertex allowed to be a nonneighbour of vertices in A'''_{12}. Thus we have our minor.

Given these results, we can now conclude that a $K_{2,t}$ minor is present in small graphs with the required number of edges.

Theorem 10 *Let t and d be positive integers, with $d < \max\left\{4, \frac{1}{11}t^{1/4}\right\}$. Let G be a graph of order $t+d$ with more than $\frac{t+1}{2}(|G|-1)$ edges. Then G has a $K_{2,t}$ minor.*

PROOF. If $d < 2$ the result is trivial, and if $2 \leq d \leq 3$ it is Lemma 6, so suppose $4 \leq d < \frac{1}{11}t^{1/4}$ and that the graph has no $K_{2,t}$ minor. Let G have maximum degree $t+s$. If $s > \frac{d-1}{2}$ or $s < \frac{d-3}{2}$, then let A be the set Y of Corollary 8 with X empty. Otherwise, let A be the set Y of Lemma 9 with X empty.

Now let $t+s'$ be the maximum degree in G of any vertex not in A. If $s' > \frac{d-1}{2}$ or $s' < \frac{d-3}{2}$, then let B be the set Y of Corollary 8 with $X = A$. Otherwise, let B be the set Y of Lemma 9 with $X = A$. Now A and B provide one half of the minor, and their common neighbours the other half.

4 Large graphs

For some of the proofs in this section, we use arguments involving linking. This is defined as follows:

Definition 11 *A graph G is said to be* k-linked *if $|G| \geq 2k$ and, for all distinct vertices $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$ of G, there exist vertex-disjoint paths from x_i to y_i for all i.*

We use the following result about linking, proved by Bollobás and Thomason [11].

Theorem 12 (Bollobás and Thomason [11]) *If a graph is $22k$-connected then it is k-linked.*

Lemma 13 *Let $t > 200$ be a positive integer. Let G be a graph with average degree at least $t-3$. Suppose that G has a vertex v with degree at least $t + 50(\log t)^2$, such that $G - v$ is connected. Suppose that there are at least $(t-3)/2$ triangles on every edge from v. Then G has a $K_{2,t}$ minor. Further, if $d(v) \geq \frac{5}{4}t$, then G has a $K_{2,1.03t}$ minor.*

PROOF. Let v have degree $\beta(t-3)$, where $\beta > 1$. Every neighbour of v has at least $(t-3)/2$ neighbours in common with v. Thus, if u is any neighbour of v, and w is a random neighbour of v (chosen uniformly at random from $\Gamma(v)$), we have that $\mathbb{P}(u \notin \Gamma(w)) \leq 1 - 1/2\beta$. If (for some positive integer k) w_1, w_2, $\ldots$, w_k are (not necessarily distinct) neighbours of v chosen uniformly and independently at random from $\Gamma(v)$, and we write $W = \{\, w_i : 1 \leq i \leq k \,\}$, then $\mathbb{P}(u \notin \Gamma(W)) \leq (1 - 1/2\beta)^k < \exp(-k/2\beta)$.

If $\beta < 2$, let $k = \lceil 2\beta \log(\beta(t-3)) \rceil$. Then, for each $u \in \Gamma(v)$, we have $\mathbb{P}(u \notin \Gamma(W)) < |\Gamma(v)|^{-1}$. Thus, with positive probability, all vertices of $\Gamma(v)$ are neighbours of some vertex of W. Fix some such W. If $\beta \geq 2$, let $k = 3$. Then $\mathbb{P}(u \notin \Gamma(W)) \leq (1-1/2\beta)^3 = 1 - 3/2\beta + 3/4\beta^2 - 1/8\beta^3 < 1 - 3/2\beta + 3/8\beta = 1 - 9/8\beta$. Thus, with positive probability, W has at least $\frac{9}{8}(t-3)$ neighbours in $\Gamma(v)$. Fix some such W.

$G - v$ is connected, so there are some paths in $G - v$ that connect W; clearly we may take such paths so that the path from w_i to w_j, if any, does not pass through any other element of W. Furthermore, if it contains more than one neighbour of either w_i or w_j, it may be shortened, and if it contains more than two neighbours of some other vertex $w_\ell \in W$, then it may be replaced by two paths, from w_i to w_ℓ and from w_ℓ to w_j, containing fewer interior vertices in total. Thus we arrive at a set of paths, such that the path from w_i to w_j contains at most one neighbour of each endpoint and at most two neighbours of each other element of W. There need only be $k-1$ paths to form a spanning tree. Add the interior vertices of these paths to W to form W'. Then W has at most $(k-1)(2k-2)$ neighbours in $W' \setminus W$, so at most $(k-1)(2k-2)+k < 2k^2$ neighbours in W'.

If $\beta < 2$, we now observe that $k < 5 \log t$. Thus W has at least t neighbours in $\Gamma(v) \setminus W'$, yielding our $K_{2,t}$ minor. If $\beta < 2$ but $d(v) \geq \frac{5}{4}t \geq t + 50(\log t)^2$, then $t > 19000$ and $\frac{5}{4}t - 2k^2 > 1.03t$, yielding our $K_{2,1.03t}$ minor. If $\beta \geq 2$, observe that $(t-3)/8 - 2k^2 = t/8 - 18 - \frac{3}{8} > 0.03t$, so W has at least $1.03t$ neighbours in $\Gamma(v) \setminus W'$.

If instead we had wished to find a $K_s + \overline{K_t}$ minor in the above lemma, we could have chosen $s-1$ sets of vertices similarly to the set W above, and made them connected using Theorem 12, provided that β is not too large.

Lemma 14 *Let $t > 10^8$ be a positive integer. Let G be a graph with average degree at least $t+1$, minimum degree at least $(t+1)/2$, at least $(t-1)/2$ triangles on every edge, and connectivity at least $150 \log t$. Let $|G| \geq t+300(\log t)^2$. Then G has a $K_{2,t}$ minor.*

PROOF. Let v be a vertex of maximum degree. If $d(v) \geq t + 50(\log t)^2$, the result follows by Lemma 13, so we suppose $t+1 \leq d(v) < t+50(\log t)^2$. If (with similar notation to the proof of Lemma 13) we put $d(v) = \beta(t-1)$, we have $\beta < 2$. Put $k = \lceil 2\beta \log(\beta(t-1)) \rceil$. As in that proof, choose W as k vertices taken independently at random from $\Gamma(v)$, and fix some particular W such that all vertices of $\Gamma(v)$ are neighbours of some vertex of W. (If this W happens to have fewer than k distinct vertices, add some arbitrary neighbours of v to W to make it up to k vertices.)

Now let y and z be any neighbours in $G - v - W$. Let $X = \Gamma(y) \cap \Gamma(z)$, so that $|X| \geq (t-1)/2$. Write $Y = \{v, y\}$ and $Z = W \cup \{z\}$. We wish to add some vertices to Y and Z such that each becomes connected. Enumerate Z as z_1, z_2, $\ldots$, z_{k+1}. The minimum degree of G is sufficient that for each i with $1 \leq i \leq k+1$ we may find $z_{i,1}$ and $z_{i,2}$ neighbours of z_i, all the $z_{i,j}$ being distinct and none of them being v or y. Now $k+1 < 5(\log t)^2$, so $G - Z$ is $22(k+1)$-connected, so $(k+1)$-linked, so we may find vertex-disjoint paths from v to y and from $z_{i,2}$ to $z_{i+1,1}$ for all i. This yields a path that may be added to Y to connect it, and paths that may be added to Z to connect that set.

As in the previous proof, we need to ensure that these paths consume few neighbours of the sets to which they are added. In the case of Y, the path may be shortened so that it contains at most one neighbour of each endpoint; letting the augmented set be Y', we then see that Y has at most $|Y| + 2 = 4$ neighbours within Y'. In the case of Z, we end up with at most k paths, each containing at most $2k$ neighbours of vertices of Z, and adding these paths to make a set Z', so we have that Z has at most $|Z| + 2k^2 = 2k^2 + k + 1$ neighbours within Z'.

Now, both Y' and Z' have as neighbours $\Gamma(v) \cup X$. If the graph does not have the required minor, it follows that $|\Gamma(v) \cup X| - 4 - (2k^2 + k + 1) < t$. Since $|\Gamma(v) \cup X| = |\Gamma(v)| + |X| - |\Gamma(v) \cap X| \geq \frac{3}{2}(t+1) - |\Gamma(v) \cap X|$, we must have $|\Gamma(v) \cap X| > (t-1)/2 - 2k^2 - k - 5 > d(v)/3$. But this means that every vertex of $G - v - \Gamma(v)$ has at least $d(v)/3$ neighbours in $\Gamma(v)$, so some vertex u of $\Gamma(v)$ has at least $|G - v - \Gamma(v)|/3$ neighbours in $G - v - \Gamma(v)$.

But $|G - v - \Gamma(v)| \geq 250(\log t)^2 - 1$, so $|\Gamma(v) \cup \Gamma(u) - v| > t + 80(\log t)^2$. Contracting the edge between v and u leaves a graph satisfying the conditions of Lemma 13.

To find a $K_s + \overline{K_t}$ minor above, s vertices could have been chosen in place of y and z.

Lemma 15 *Let $t > 10^{29}$ be a positive integer. Let G be a connected graph with more than $\frac{t+1}{2}(|G| - 1)$ edges. Then G has a $K_{2,t}$ minor.*

PROOF. We work by induction on $|G|$. Note that $300(\log t)^2 < \frac{1}{11}t^{1/4}$. Thus, if $|G| < t + \frac{1}{11}t^{1/4}$, the result follows by Theorem 10, and otherwise we have $|G| \geq t + \frac{1}{11}t^{1/4} > t + 300(\log t)^2$.

If G has a vertex with degree less than or equal to $(t+1)/2$, remove it; if G has an edge on which there are fewer than $t/2$ triangles, contract it. These operations pass from G to a minor of G with fewer vertices, and do not decrease $e(G) - \frac{t+1}{2}|G|$. Thus we may suppose that G has minimum degree at least $(t+1)/2 + 1$ and at least $t/2$ triangles on every edge.

If $\kappa(G) \geq 150 \log t$, we are done by Lemma 14, so suppose $\kappa(G) < 150 \log t$. Let S be a cutset with $|S| = \kappa(G)$.

If $\kappa(G) = 1$, let X be some component of $G - S$. Both $G - X$ and $G[X \cup S]$ are minors of G with fewer vertices; if neither satisfies the conditions of the theorem, observe that together they have $e(G)$ edges, so that $e(G) \leq \frac{t+1}{2}(|G - X| - 1 + |X \cup S| - 1) = \frac{t+1}{2}(|G| - 1)$, a contradiction. Thus one of $G - X$ and $G[X \cup S]$ satisfies the conditions of the theorem.

It remains to consider the case of $2 \leq \kappa(G) < 150 \log t$. In this case, we may assume that $\Delta(G) < t + 50(\log t)^2$, since otherwise we may apply Lemma 13. If X is any component of $G - S$, and neither $G - X$ nor $G[X \cup S]$ satisfy the conditions of the theorem, we must have $e(G - X) \leq \frac{t+1}{2}(|G - X| - 1)$ and $e(G[X \cup S]) \leq \frac{t+1}{2}(|X \cup S| - 1)$. But then

$$\begin{aligned} e(G[X]) &\geq e(G) - e(G - X) - |S||X| \\ &\geq \tfrac{t+1}{2}(|G| - 1) + 1 - \tfrac{t+1}{2}(|G - X| - 1) - |S||X| \\ &= (\tfrac{t+1}{2} - |S|)|X| + 1 \\ &> (\tfrac{t+1}{2} - 150 \log t)|X|, \end{aligned}$$

for all components X of $G - S$. Each graph $G[X]$ must also have minimum degree at least $(t+1)/2 - 150 \log t$ and at least $t/2 - 150 \log t$ triangles on every edge.

Now let u and v be two vertices of S. We will find disjoint subsets U and V of X such that $G[U \cup \{u\}]$ and $G[V \cup \{v\}]$ are connected and $U \cup \{u\}$ and $V \cup \{v\}$ have at least $t/2$ common neighbours in $X - U - V$. Since $G - S$ has at least two components, we can then do the same with another component (with the same u and v) to find our minor.

Suppose x and $y \in X$ are neighbours. Then, by 2-connectivity, $G[X \cup \{u, v\}]$ has two vertex-disjoint paths from $\{u, v\}$ to $\{x, y\}$; without loss of generality, suppose that these paths are from u to x and from v to y. The path from u to x may be supposed to contain just one neighbour of x; that from v to y may be supposed to contain just one neighbour of y. Suppose we put the path from u to x in U, and that from v to y in V. Consider the common neighbours of x and y in X. At most one is in U, and at most one is in V. If they have as many as $t/2+2$ common neighbours in X, we have our minor, so suppose that $|\Gamma_X(x) \cap \Gamma_X(y)| \leq t/2+1$. This argument applies for any pair of neighbours in X, so we may suppose this inequality applies for all such pairs of neighbours.

If $|\Gamma_X(x)|+|\Gamma_X(y)| > 15t/8$, then $|\Gamma_X(x) \cup \Gamma_X(y) \setminus \{x, y\}| = |\Gamma_X(x)|+|\Gamma_X(y)| - |\Gamma_X(x) \cap \Gamma_X(y)| - 2 > 11t/8 - 3$. Contracting the edge xy, and contracting all components of $G - S$ other than X into S, we may then apply Lemma 13 to find a $K_{2,1.03(t-1000\log t)}$ minor. Since $1.03(t - 1000\log t) > t$, we may now suppose that $|\Gamma_X(x)| + |\Gamma_X(y)| \leq 15t/8$ for all x, y neighbours in X.

$G[X]$ has average degree at least $t+1-300\log t$; that is, $2e(G[X]) = \sum_{x \in X} d_X(x) \geq (t+1-300\log t)|X|$. It follows that $\sum_{x \in X} d_X(x)^2 \geq (t+1-300\log t)(2e(G[X]))$. But $\sum_{x \in X} d_X(x)^2 = \frac{1}{2}\sum_{x \in X}\sum_{y \in \Gamma_X(x)}(d_X(x) + d_X(y)) \leq \frac{15}{16}\sum_{x \in X}\sum_{y \in \Gamma_X(x)} t = \frac{15}{16}t(2e(G[X]))$, a contradiction given the lower bound on t.

To find a $K_s + \overline{K_t}$ minor above in an s-connected graph, paths would be taken from the cutset to more vertices than just x and y.

Of course, if G is not connected, we may just take some connected component of G with sufficiently many edges. We thus derive Theorem 2, which (considering the lower bound of Theorem 4) is best possible for $|G| \equiv 1 \pmod{t}$.

5 The case $s > 2$

We now return to the more general problem of $K_{s,t}$ and $K_s + \overline{K_t}$ minors. Even for $s = 3$, we have no results better than the average degree $O\left(t\sqrt{\log t}\right)$ that forces a K_{s+t} minor, and so a $K_{s,t}$ minor. None of the methods of Section 3 apply to these more general minors. Many of the methods of Section 4 do apply more generally—as described there, in many cases extra vertices can be taken and enough paths found using Theorem 12—but significant extra arguments would be needed to obtain useful results this way. For example, Lemma 13 can readily be extended if β is small, but when β is large there seems to be no simple way to apply it to $K_{s,t}$ minors for $s > 2$.

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